1. Curvature of left invariant Riemannian metrics on Lie groups

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1.1 Introduction

This course is based on the survey paper of Milnor [13] which has became a classic. As its title suggests, this paper deals with the curvature properties of the large class of so called Riemannian Lie groups, i.e., Lie groups endowed with left invariant Riemannian metrics. Milnor's paper gave many known results on the topic, proved and conjectured many new ones. This triggered a huge interest among the community of mathematicians and, since, many conjectures in the paper has been proved and the research on the topic is still going on. It is a well established fact that there is deep relations between the topology and the geometry of a manifold in one side, and the curvature of a given Riemannian metric on this manifold on the other side. There is a long list of theorems supporting this fact (see for instance [4]). When the manifold is a Lie group and the metric is left invariant the curvature is also strongly related to the group's structure or equivalently to the Lie algebra's structure. The main purpose of this course is to illustrate this fact by giving a detailed and self-contained exposition (based on [13]) of some results on this subject according to the spirit of this school which is using algebra to study geometric problems. I will deliberately avoide the use of some strong geometric theorems a part for Meyer's Theorem which is used in the proof of Theorem 1.5.2. Although the content is mainly the same as [13], I will adopt a different approach to establish basic formulas and to prove some results. For instance, the proof of Theorem 1.4.2 is completely different from the original proof by Milnor. There are some results which did not appear in Milnor's paper, namely, Proposition 1.4.3, Theorem 1.6.1 and Theorem 1.5.4 proved in [6].

A part from the introduction, the course is divided into six sections. In the first one we recall the basic tools in Riemannian geometry. The second one is devoted to the preliminaries properties of left invariant Riemannian metrics on Lie groups. Namely, we establish the formulas giving different curvatures at the level of the associated Lie algebras. We study also the particular case of bi-invariant Riemannian metrics. In the third section, we study Riemannian Lie groups with strictly negative, null or strictly positive sectional curvature. The same thing is done in the fourth and the fifth sections for the Ricci curvature and the scalar curvature, respectively. In the last section, we give a complete description of the class of 3-dimensional Riemannian Lie groups.

1.2 Basic tools on Riemannian manifolds

1.2.1 Riemannian metrics and Levi-Civita connection

A **Riemannian metric** on a smooth manifold M is a map which associate to any point $p \in M$ a scalar product $\langle\ ,\ \rangle_p$ on T_pM such that, for any local coordinates system $(x_1, \ldots, x_n)$ on an open set U, the local functions $g_{ij} : U \longrightarrow \mathbb{R}$ given by

$$g_{ij} = \langle \partial_{x_i}, \partial_{x_j} \rangle \tag{1.1}$$

are smooth for any $i, j = 1, \ldots, \dim M = n$. A smooth manifold with a Riemannian metric is called a **Riemannian manifold.**

Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold and $(x_1, \ldots, x_n)$ a local coordinates system. The local expression of $\langle\ ,\ \rangle$ is given by

$$\langle\ ,\ \rangle = \sum_{i,j} g_{ij} dx_i dx_j, \tag{1.2}$$

where $g_{ij} =$ are given by (1.1) and

$$dx_i dx_j = \frac{1}{2}(dx_i \otimes dx_j + dx_j \otimes dx_i).$$

Proposition 1.2.1 *Any smooth manifold carries a Riemann metric.*

Proof. Choose a locally finite open covering $\mathbb{U} = \{U_\alpha\}_{\alpha \in A}$ of M by chart domains and a subordinate partition of the unity $(f_\alpha : M \longrightarrow [0,1])_{\alpha \in A}$. For any $\alpha \in A$ define on U_α a Riemannian metric $\langle\ ,\ \rangle_\alpha$ by putting

$$\langle\ ,\ \rangle_\alpha = \sum_{i=1}^{n} (dx_i)^2.$$

Now define $\langle\ ,\ \rangle$ on M by putting, for any $p \in M$ and any $u, v \in T_pM$,

$$\langle u, v \rangle = \sum_{\alpha \in A} f_\alpha \langle u, v \rangle_\alpha. \tag{1.3}$$

One can see easily that $\langle\ ,\ \rangle$ is a Riemannian metric on M. □

Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold. For any $u \in TM$, put

$$\|u\| = \sqrt{\langle u, u \rangle}.$$

For any smooth curve $\gamma : [a,b] \longrightarrow M$, the length of γ is given by

$$\ell(\gamma) = \int_a^b \|\gamma'(t)\| dt.$$

For any $p, q \in M$, put

$$d(p,q) = \inf_{\gamma \in \mathcal{C}_{pq}} \ell(\gamma),$$

where $\mathcal{C}_{pq}$ is the set of all curve γ joining p to q.

Proposition 1.2.2 *(M,d) is a metric space.*

Proof. It is obvious that d satisfies:

1. $d(p,q) \geq 0$,
2. $d(p,q) = d(q,p)$,
3. $d(p,q) \leq d(p,r) + d(r,q)$.

To achieve the proof we need to show that if $p \neq q$ then $d(p,q) > 0$. Fix a point p and a chart $\phi : U \longrightarrow \mathbb{R}^n$ around p. Then there exists $\delta > 0$ such that $\phi^{-1}(B(\phi(p),\delta)) \subset U$ where $B(\phi(p),\delta)$ is the Euclidean ball of center $\phi(p)$ with radius δ. There exists $\lambda > 0$ such that for any $\xi \in T\phi^{-1}(B(\phi(p),\delta))$,

$$\|\xi\| \geq \lambda \sqrt{\sum_{i=1}^n \xi_i^2},$$

where $\xi = \sum \xi_i \partial_{x_i}$. Therefore, for $q \in \phi^{-1}(B(\phi(p),\delta))$,

$$d(p,q) \geq \lambda |\phi(p) - \phi(q)|.$$

For $q \in M \setminus \phi^{-1}(B(\phi(p),\delta))$, we have obviously

$$d(p,q) \geq \lambda\delta,$$

which achieves the proof. □

A linear connection ∇ on a Riemannian manifold $(M, \langle\ ,\ \rangle)$ is called compatible with the Riemannian structure if, for any smooth curve $\gamma : I \longrightarrow M$ and for any $t_0, t_1 \in I$, the parallel transport $\tau_{t_0,t_1} : T_{\gamma(t_0)}M \longrightarrow T_{\gamma(t_1)}M$ preserves the scalar product, i.e.,

$$\langle \tau_{t_0,t_1}(u), \tau_{t_0,t_1}(v)\rangle = \langle u, v\rangle, \quad u, v \in T_{\gamma(t_0)}M.$$

Proposition 1.2.3 *Let $(M, \langle\ ,\ \rangle)$ be either a Riemannian manifold and ∇ a linear connection. The following assertions are equivalent:*

1. *∇ is compatible with $\langle\ ,\ \rangle$.*
2. *For any $\gamma : I \longrightarrow M$ and any couple of vector field V, W along γ,*
$$\frac{d}{dt}\langle V(t), W(t)\rangle = \langle D_\gamma V(t), W(t)\rangle + \langle V(t), D_\gamma W(t)\rangle.$$
3. *For any vector field X, Y, Z on M,*
$$\nabla_X\langle Y, Z\rangle := X.\langle Y, Z\rangle - \langle \nabla_X Y, Z\rangle - \langle Y, \nabla_X Z\rangle = 0.$$

Proof. 1. $\Longrightarrow$ 2. Fix a smooth curve $\gamma : I \longrightarrow M$. The fact that ∇ is compatible with $\langle\ ,\ \rangle$ implies that, for any V, W parallel along γ, we have

$$\frac{d}{dt}\langle V(t), W(t)\rangle = 0.$$

Choose a family of vector fields parallel along γ say $(E_1, \ldots, E_n)$ such that for any $t \in I$, $(E_1(t), \ldots, E_n(t))$ is a basis of $T_{\gamma(t)}M$. So if V and W are two vector fields along γ we have

$$V = \sum_{i=1}^{n} V_i E_i \quad \text{and} \quad W = \sum_{i=1}^{n} W_i E_i,$$

where $V_i, W_i \in C^\infty(I, \mathbb{R})$. Thus

$$D_\gamma V = \sum_{i=1}^{n} V_i' E_i \quad \text{and} \quad D_\gamma W = \sum_{i=1}^{n} W_i' E_i,$$

Then

$$\frac{d}{dt}\langle V(t), W(t)\rangle \;=\; \sum_{i,j=1}^{n}\left(V_i'(t)W_j(t)+V_i(t)W_j'(t)\right)\langle E_i, E_j\rangle,$$

and the desired formula holds.

2. $\Longrightarrow$ 3. Fix a point $p \in M$ and denote by $\gamma : [0, \epsilon[$ the integral curve of X passing through p. Thus

$$\begin{aligned} X.\langle Y, Z\rangle(p) &= \frac{d}{dt}_{|t=0}\langle Y(\gamma(t)), Z(\gamma(t))\rangle \\ &= \langle (D_\gamma Y)(\gamma(0)), Z(p)\rangle + \langle Y(p), (D_\gamma Y)(\gamma(0))\rangle \\ &= \langle \nabla_X Y(p), Z(p)\rangle + \langle Y(p), \nabla_X Z(p)\rangle, \end{aligned}$$

and the formula follows.

3. $\Longrightarrow$ 1. Exercise. □

Levi-Civita connection The following theorem is a fundamental result in Riemannian geometry.

Theorem 1.2.1 *For any Riemannian manifold $(M, \langle\ ,\ \rangle)$ there exists a unique linear connection torsion free and compatible with the metric.*
This connection is called Levi-Civita connection.

Proof. Since $\langle\ ,\ \rangle$ is nondegenerate, to compute $\nabla_X Y$ it suffices to compute $\langle \nabla_X Y, Z\rangle$. By using repeatedly the fact that ∇ is compatible with the metric and torsion free, we get

$$\begin{aligned}\langle\nabla_X Y, Z\rangle &= X.\langle Y, Z\rangle - \langle Y, \nabla_X Z\rangle \\ &= X.\langle Y, Z\rangle - \langle Y, \nabla_Z X\rangle - \langle Y, [X, Z]\rangle \\ &= X.\langle Y, Z\rangle - Z.\langle Y, X\rangle + \langle\nabla_Z Y, X\rangle - \langle Y, [X, Z]\rangle \\ &= X.\langle Y, Z\rangle - Z.\langle Y, X\rangle + \langle\nabla_Y Z, X\rangle + \langle[Z, Y], X\rangle \\ &\quad -\langle Y, [X, Z]\rangle \\ &= X.\langle Y, Z\rangle - Z.\langle Y, X\rangle + Y.\langle Z, X\rangle - \langle Z, \nabla_Y X\rangle \\ &\quad +\langle[Z, Y], X\rangle - \langle Y, [X, Z]\rangle \\ &= X.\langle Y, Z\rangle - Z.\langle Y, X\rangle + Y.\langle Z, X\rangle - \langle Z, \nabla_X Y\rangle \\ &\quad +\langle Z, [X, Y]\rangle + \langle[Z, X], Y\rangle + \langle[Z, Y], X\rangle.\end{aligned}$$

Thus

$$\begin{aligned}2\langle\nabla_X Y, Z\rangle &= X.\langle Y, Z\rangle + Y.\langle X, Z\rangle - Z.\langle X, Y\rangle \\ &\quad +\langle[X, Y], Z\rangle + \langle[Z, X], Y\rangle + \langle[Z, Y], X\rangle. \qquad (1.4)\end{aligned}$$

This formula gives the uniqueness and can be used to define ∇ and to check that ∇ satisfies the required properties. □

The formula (1.4) is called Koszul's formula. Let use it to compute the Christoffel symbols of Levi-Civita connection. Let $(x_1, \ldots, x_n)$ be a coordinates system. The Christoffel symbols are given by

$$\nabla_{\partial_i}\partial_j = \sum_{l=1}^{n} \Gamma_{ij}^l \partial_l.$$

According to (1.4), we have, for any $i, j, l = 1 \ldots, n$,

$$2\langle\nabla_{\partial_i}\partial_j, \partial_l\rangle = \partial_i.g_{jl} + \partial_j.g_{il} - \partial_l.g_{ij},$$

where $g_{ij} =< \partial_i, \partial_j >$. So

$$\sum_{k=1}^{n} g_{kl}\Gamma_{ij}^{k} = \frac{1}{2}\left(\partial_i.g_{jl} + \partial_j.g_{il} - \partial_l.g_{ij}\right).$$

Denote by $G = (g_{ij})_{1\leq i,j\leq n}$ and its inverse by $G^{-1} = (g^{ij})_{1\leq i,j\leq n}$, we get

$$\Gamma_{ij}^{k} = \frac{1}{2}\sum_{l=1}^{n} g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right). \tag{1.5}$$

1.2.2 Curvature of Riemannian metrics

Theorem 1.2.2 *Let $(M, \langle\ ,\ \rangle)$ be a n-dimensional Riemannian manifold and ∇ its Levi-Civita connection. Then the map*

$$R : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$

given by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - (\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z) \tag{1.6}$$

is a tensor field of type $(3,1)$. Moreover, R is the unique tensor field satisfying for any variation $(s,t) \mapsto \Gamma(s,t) \in M$ and any vector field along Γ

$$D_S D_T Y - D_T D_S Y = -R(S,T)Y. \tag{1.7}$$

Proof. We will show three assertions:

1. R given by (1.6) is a tensor field, i.e., it is $C^\infty(M)$ 3-linear.

2. R satisfies (1.7).

3. If R' satisfies (1.7) then $R' = R$.

Let $f \in C^\infty(M)$. we have

$$\begin{aligned}
-R(fX,Y)Z &= \nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]}Z \\
&= f\nabla_X\nabla_Y Z - \nabla_Y f\nabla_X Z - \nabla_{f[X,Y]-Y(f)X}Z \\
&= f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - Y(f)\nabla_X Z - f\nabla_{[X,Y]}Z + Y(f)\nabla_X Z \\
&= -fR(X,Y)Z.
\end{aligned}$$

Since $R(X,Y)Z = -R(Y,X)Z$, we deduce immediately that $R(X,Y)Z = fR(X,Y)Z$.

On the other hand, we have

$$\begin{aligned}
-R(X,Y)fZ &= \nabla_X\nabla_Y fZ - \nabla_Y\nabla_X fZ - \nabla_{[X,Y]}fZ \\
&= \nabla_X(f\nabla_Y Z + Y(f)Z) - \nabla_Y(f\nabla_X Z + X(f)Z) - f\nabla_{[X,Y]}Z - [X,Y](f)Z \\
&= f\nabla_X\nabla_Y Z + X(f)\nabla_Y Z + Y(f)\nabla_X Z + X(Y(f))Z - f\nabla_Y\nabla_X Z \\
&\quad -Y(f)\nabla_X Z - X(f)\nabla_Y Z - Y(X(f))Z - f\nabla_{[X,Y]}Z - [X,Y](f)Z \\
&= -fR(X,Y)Z,
\end{aligned}$$

so we have shown the first assertion.

Let $(x_1,\dots,x_n)$ be a coordinates system and put $\Gamma(s,t) = (x_1(s,t),\dots,x_n(s,t))$ and $Y = \sum_{i=1}^n Y_i(s,t)\partial_i$. We get

$$T(s,t) = \sum_{i=1}^n \frac{\partial x_i}{\partial t}\partial_i \quad \text{and} \quad S(s,t) = \sum_{i=1}^n \frac{\partial x_i}{\partial s}\partial_i.$$

We have also

$$\begin{aligned}
D_T Y &= \sum_{i=1}^n \frac{\partial Y_i}{\partial t}\partial_i + \sum_{i,j=1}^n Y_i \frac{\partial x_j}{\partial t}\nabla_{\partial_j}\partial_i \\
D_S D_T Y &= \sum_{i=1}^n \frac{\partial^2 Y_i}{\partial s \partial t}\partial_i + \sum_{i,j=1}^n \left\{ \frac{\partial Y_i}{\partial t}\frac{\partial x_j}{\partial s} + \frac{\partial Y_i}{\partial s}\frac{\partial x_j}{\partial t} + Y_i \frac{\partial^2 x_j}{\partial s \partial t} \right\} \nabla_{\partial_j}\partial_i \\
&\quad + \sum_{i,j,k} Y_i \frac{\partial x_j}{\partial t}\frac{\partial x_k}{\partial s}(\nabla_{\partial_k}\nabla_{\partial_j}\partial_i).
\end{aligned}$$

The expression of $D_T D_S Y$ is similar, one needs just to convert the roles of s and t. So we get

$$\begin{aligned}(D_S D_T - D_T D_S)Y &= \sum_{i,j,k} Y_i \frac{\partial x_j}{\partial t}\frac{\partial x_k}{\partial s}\left(\nabla_{\partial_k}\nabla_{\partial_j}\partial_i - \nabla_{\partial_j}\nabla_{\partial_k}\partial_i\right) \\ &= \sum_{i,j,k} Y_i \frac{\partial x_j}{\partial t}\frac{\partial x_k}{\partial s} R(\partial_k, \partial_j)\partial_i = -R(S,T)Y,\end{aligned}$$

which achieves the proof of the second assertion.

Let R' be a $(3,1)$-tensor field satisfying (1.7). Let $p \in M$ and $\phi = (x_1, \ldots, x_n)$ a coordinates system around p satisfying $\phi(p) = 0$. For $1 \leq i < j \leq n$ fixed, we consider the variation Γ given by

$$\Gamma(s,t) = \phi^{-1}(0, \ldots, s, \ldots, t, \ldots, 0),$$

s is at the i-place and t at the j-place. We have $S = \partial_i$ and $T = \partial_j$. Take $Y = \partial_k$. Since R' satisfies (1.7), we get

$$R'(\partial_i, \partial_j)\partial_k = -(D_S D_T - D_T D_S)Y = R(\partial_i, \partial_j)\partial_k$$

thus $R' = R$ and the proof of the theorem is given. □

Let us express the curvature tensor in a coordinates system $(x_1, \ldots, x_n)$. Since $\mathbb{R}$ is linear in its three entries, there exists locale functions R^l_{ijk} such that

$$R(\partial_i, \partial_j)\partial_k = \sum_{l=1}^{n} R^l_{ijk}\partial_l.$$

Let us compute them in function of Christoffel symbols. We have

$$\nabla_{\partial_j}\partial_k = \sum_{l=1}^{n} \Gamma^l_{jk}\partial_l$$

which give

$$\nabla_{\partial_i}\nabla_{\partial_j}\partial_k = \sum_{l=1}^{n} \partial_i(\Gamma_{jk}^l)\partial_l + \sum_{l,m} \Gamma_{jk}^l \Gamma_{il}^m \partial_m.$$

By gathering these terms, we get

$$\nabla_{\partial_i}\nabla_{\partial_j}\partial_k = \sum_{l=1}^{n} \left(\partial_i(\Gamma_{jk}^l) + \sum_{m=1}^{n} \Gamma_{jk}^m \Gamma_{im}^l \right) \partial_l.$$

Since $[\partial_i, \partial_j] = 0$, we get finally

$$R_{ijk}^l = \partial_j(\Gamma_{ik}^l) - \partial_i(\Gamma_{jk}^l) + \sum_{m=1}^{n} \left(\Gamma_{ik}^m \Gamma_{jm}^l - \Gamma_{jk}^m \Gamma_{im}^l \right). \tag{1.8}$$

The following proposition summarizes the algebraic properties of the curvature tensor.

Proposition 1.2.4 *Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold. For any $p \in M$ and any $x, y, z, w \in T_pM$, we have*

1. $R(x, y)z = -R(y, x)z$ *(skew-symmetry);*
2. $R(x, y)z + R(y, z)x + R(z, x)y = 0$ *(Bianchi's identity);*
3. $\langle R(x, y)z, w\rangle = -\langle R(x, y)w, z\rangle$*;*
4. $\langle R(x, y)z, w\rangle = \langle R(z, w)x, y\rangle$.

Proof. The relation 1. and 2. are multi-linear en x, y, z so it suffices to establish them for $\partial_i, \partial_j, \partial_k$ associated to a local coordinates system $(x_1, \ldots, x_n)$. The relation 1. is immediate. We have

$$\begin{aligned}
&-R(\partial_i, \partial_j)\partial_k - R(\partial_j, \partial_k)\partial_i - R(\partial_k, \partial_i)\partial_j = \\
&\left(\nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k\right) + \left(\nabla_{\partial_j}\nabla_{\partial_k}\partial_i - \nabla_{\partial_k}\nabla_{\partial_j}\partial_i\right) + \left(\nabla_{\partial_k}\nabla_{\partial_i}\partial_j - \nabla_{\partial_i}\nabla_{\partial_k}\partial_j\right) = \\
&\nabla_{\partial_i}\left(\nabla_{\partial_j}\partial_k - \nabla_{\partial_k}\partial_j\right) - \nabla_{\partial_j}\left(\nabla_{\partial_i}\partial_k - \nabla_{\partial_k}\partial_i\right) + \nabla_{\partial_k}\left(\nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i\right) = 0,
\end{aligned}$$

since $\nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i = [\partial_i, \partial_j] = 0$.
For the relation 3. we consider a family of vector fields X, Y, Z, W extending

x, y, z, w. We have, since ∇ preserves the Riemannian metric,

$$\begin{aligned}
\langle \nabla_X \nabla_Y Z, W\rangle &= X.\langle \nabla_Y Z, W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle \\
&= XY.\langle Z, W\rangle - X.\langle Z, \nabla_Y W\rangle - Y.\langle Z, \nabla_X W\rangle \\
&\quad + \langle Z, \nabla_Y \nabla_X W\rangle, \\
\langle \nabla_Y \nabla_X Z, W\rangle &= YX.\langle Z, W\rangle - Y.\langle Z, \nabla_X W\rangle - X.\langle Z, \nabla_Y W\rangle \\
&\quad + \langle Z, \nabla_X \nabla_Y W\rangle, \\
\langle \nabla_{[X,Y]} Z, W\rangle &= (XY - XY).\langle Z, W\rangle - \langle Z, \nabla_{[X,Y]} W\rangle.
\end{aligned}$$

We deduce by using (1.6) that

$$\langle R(X,Y)Z, W\rangle = -\langle Z, R(X,Y)W\rangle,$$

which gives the third formula.
The fourth formula is a purely algebraic consequence of the three first relations. We will establish it in the following lemma. □

Lemma 1.2.1 *Let $r : V^4 \longrightarrow \mathbb{R}$ be a 4-linear map on a real vector space V satisfying*

1. $r(x, y, z, w) = -r(y, x, z, w)$;
2. $r(x, y, z, w) + r(y, z, x, w) + r(z, x, y, w) = 0$;
3. $r(x, y, z, w) = -r(x, y, w, z)$.

Then r satisfies

$$r(x, y, z, w) = r(z, w, x, y). \tag{1.9}$$

Proof. We will get this relation by permuting the second relation in the following way:

$$\begin{aligned}
0 &= r(x, y, z, w) + r(y, z, x, w) + r(z, x, y, w) \\
0 &= r(y, z, w, x) + r(z, w, y, x) + r(w, y, z, x) \\
0 &= -r(z, w, x, y) - r(w, x, z, y) - r(x, z, w, y) \\
0 &= -r(w, x, y, z) - r(x, y, w, z) - r(y, w, x, z)
\end{aligned}$$

and by using the first and third relation. □

A 4-linear form satisfying the hypothesis of Lemma 1.2.1 is called of curvature type.

Example 1 *Let V be a real vector space endowed with a scalar product $\langle\ ,\ \rangle$. Then the map $\rho : V^4 \longrightarrow \mathbb{R}$ given by*

$$\rho(x,y,z,w) = \begin{vmatrix} \langle x,w\rangle & \langle x,z\rangle \\ \langle y,w\rangle & \langle y,z\rangle \end{vmatrix}$$

is of curvature type.

Definition 1.2.1 *Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold. For any $p \in M$ and for any couple of linearly independent tangent vectors $u, v \in T_pM$, put*

$$Q(u,v) = \frac{\langle R(u,v)u, v\rangle}{|u|^2|v|^2 - \langle u,v\rangle^2}.$$

The map Q is called sectional curvature of M.

Proposition 1.2.5 *The sectional curvature $Q(u,v)$ depends only on the plan spanned by u, v.*

Proof. It is a consequence of the following lemma. □

Lemma 1.2.2 *Let $r, \rho : V^4 \longrightarrow \mathbb{R}$ be two 4-linear forms of curvature type. If $P = \mathrm{span}\{x,y\} = \mathrm{span}\{x',y'\}$ and $Q = \mathrm{span}\{z,w\} = \mathrm{span}\{z',w'\}$ where P and Q are two plans in V. Then*

$$\frac{r(x,y,z,w)}{\rho(x,y,z,w)} = \frac{r(x',y',z',w')}{\rho(x',y',z',w')}.$$

Proof. Exercise. □

The curvature tensor gives the sectional curvatures. The convert is also true.

Proposition 1.2.6 *The curvature tensor is entirely determined by the sectional curvatures.*

Proof. The formula which gives R in function of K is given in the following lemma. $\square$

Lemma 1.2.3 *Let* $r : V^4 \longrightarrow \mathbb{R}$ *be of curvature type and let* s *given by* $s(x,y) = r(x,y,y,x)$. *Then*

$$\begin{aligned}24r(x,y,z,w) &= s(x+w,y+w) - s(x-w,y+z)\\ &\quad -s(x+w,y-z) - s(x-w,y-z)\\ &\quad -s(y+w,x+z) - s(y-w,x+z)\\ &\quad +s(y+w,x-z) - s(y-w,x-z).\end{aligned}$$

Proof. We find first an intermediary relation:

$$\begin{aligned}& r(x,y+z,y+z,w) - r(x,y-z,y-z,w) - r(y,x+z,x+z,w) + r(y,x-z,x-z,w)\\ =\ & 2r(x,y,z,w) + 2r(x,z,y,w) - 2r(y,x,z,w) - 2r(y,z,x,w)\\ =\ & 4r(x,y,z,w) + 2r(x,z,y,w) + 2r(z,y,x,w)\\ =\ & 6r(x,y,z,w) + 2r(y,x,z,w) + 2r(x,z,y,w) + 2r(z,y,x,w)\\ =\ & 6r(x,y,z,w).\end{aligned}$$

Therefore we use

$$\begin{aligned}r(u+w,v,v,u+w) - r(u-w,v,v,u-w) &= 2r(u,v,v,w) + 2r(w,v,v,u)\\ &= 4r(u,v,v,w).\end{aligned}$$

$\square$

A Riemannian manifold has a positive (resp. negative) sectional curvature if for any $p \in M$ and for any plan $P \subset T_pM$, $Q(P) \geq 0$ (resp. $Q(P) \leq 0$).
A Riemannian manifold has a strictly positive (resp. strictly negative) sectional curvature if for any $p \in M$ and for any plan $P \subset T_pM$, $Q(P) > 0$ (resp. $Q(P) < 0$).
A Riemannian manifold has a constant sectional curvature κ if for any $p \in M$ and for any plan $P \subset T_pM$, $Q(P) = \kappa$.

Proposition 1.2.7 *$(M, \langle\ ,\ \rangle)$ is of constant sectional curvature κ if and only if the curvature tensor is given by*

$$R(x,y)z = \kappa(\langle x,z\rangle y - \langle y,z\rangle x).$$

Proof. One can see easily that if the curvature tensor is given by the formula above then the sectional is constant. For the converse, put

$$r(x,y,z,w) = \langle R(x,y)z,w\rangle - \kappa(\langle x,z\rangle\langle y,w\rangle - \langle y,z\rangle\langle x,w\rangle).$$

Then r is of curvature type and for any x, y, $r(x,y,x,y)$ =constante= 0. Then by lemma 1.2.3 $r = 0$ and the lemma follows. □

1.2.3 Ricci curvature, scalar curvature and Killing vector fields

Let $(M, \langle\ ,\ \rangle)$ be a Riemannian metric and R its curvature tensor. The Ricci curvature is the trace of R, namely, if $(e_1, \ldots, e_n)$ is an orthonormal basis of T_pM, we have for any $u, v \in T_pM$,

$$\begin{aligned} \mathrm{ric}(u,v) &= \mathrm{tr}(x \mapsto R(u,x)v) \\ &= \sum_{i=1}^{n} \langle R(u,e_i)v, e_i\rangle \\ &= \sum_{i=1}^{n} \langle R(v,e_i)u, e_i\rangle \quad \text{(see Proposition 1.2.4)} \\ &= \mathrm{ric}(v,u). \end{aligned} \tag{1.10}$$

We can see the Ricci curvature as an endomorphism $\mathrm{Ric}_p : T_pM \longrightarrow T_pM$ via the formula

$$\mathrm{ric}(u,v) = \langle \mathrm{Ric}_p(u), v\rangle = \langle \mathrm{Ric}_p(v), u\rangle, \quad \forall u, v \in T_pM. \tag{1.11}$$

From (1.10), we have

$$\mathrm{Ric}_p(u) = \sum_{i=1}^{n} R(u,e_i)e_i. \tag{1.12}$$

We call Ric Ricci operator. It is a symmetric field of endomorphism and hence has real eigenvalues in each point $p \in M$ say $\lambda_1(p) \leq \ldots \leq \lambda_n(p)$. The metric has positive (resp. strictly positive) Ricci curvature if, for any $i = 1, \ldots, n$, $\lambda_i(p) \geq 0$ (resp. $\lambda_i(p) > 0$). In analogue way, we can define metric of negative Ricci curvature and strictly negative curvature. The metric is called Einstein if $\mathrm{ric} = \lambda\langle\ ,\ \rangle$ where λ is a constant.

The scalar curvature of $(M, \langle\ ,\ \rangle)$ is the C^∞ function $\mathfrak{s} : M \longrightarrow \mathbb{R}$ given by

$$\mathfrak{s}(p) = \mathrm{trRic}_p = \sum_{i=1}^{n} \lambda_i(p) = \sum_{i=1}^{n} \mathrm{ric}(e_i, e_i). \tag{1.13}$$

A Killing vector field of $(M, \langle\ ,\ \rangle)$ is a vector field X such that its local flow is consisting of isometries of $\langle\ ,\ \rangle$. This is equivalent to the Lie derivative of $\langle\ ,\ \rangle$ along X vanishes, i.e.,

$$L_X(\langle\ ,\ \rangle)(Y, Z) = X.\langle Y, Z\rangle - \langle [X, Y], Z\rangle - \langle [X, Z], Y\rangle = 0, \tag{1.14}$$

for any $Y, Z \in \mathcal{X}(M)$. It is obvious that the space of Killing vector fields $\mathrm{Kill}(\langle\ ,\ \rangle)$ is a real Lie algebra for the Lie bracket. It is a non obvious fact (see [11]) that, actually when the metric is geodesically complete $\mathrm{Kill}(\langle\ ,\ \rangle)$ is a finite dimensional Lie algebra.

We end this section by stating Meyer's Theorem. For the proof, one can see [12] pp. 201.

Theorem 1.2.3 *Suppose M is a complete, connected Riemannian n-manifold whose Ricci tensor satisfies the following inequality for any $u \in TM$:*

$$\mathrm{ric}(u, u) \geq \frac{n-1}{\rho^2}|u|^2.$$

Then M is compact, with finite dimensional fundamental group, and diameter at most $\pi\rho$.

1.3 Left invariant Riemannian metrics on Lie groups

1.3.1 Definition and basic properties

Let G be a n-dimensional Lie group, $\mathcal{X}^\ell(G)$ the Lie algebra of left invariant vector fields on G and $\mathfrak{g} = T_eG$ where e stands for the neutral element of G. For any $u \in \mathfrak{g}$, we denote by $u^\ell \in \mathcal{X}^\ell(G)$ so that

$$u^\ell(a) = T_e\mathcal{L}_a(u), \tag{1.15}$$

where $T_e\mathcal{L}_a : \mathfrak{g} \longrightarrow T_aG$ is the tangent application at e for the left multiplication by a, $\mathcal{L}_a : G \longrightarrow G$, $b \mapsto ab$. It is obvious that the map $u \mapsto u^\ell$ is an isomorphism of real vector spaces between $\mathfrak{g}$ and $\mathcal{X}^\ell(G)$. Thus $\mathfrak{g}$ carries a Lie bracket $[\ ,\ ]_\mathfrak{g}$ given by

$$[u, v]_\mathfrak{g} = [u^\ell, v^\ell](e), \quad \forall u, v \in \mathfrak{g}.$$

A left invariant Riemannian metric on G is a Riemannian metric h on G such that for any $a \in G$, $\mathcal{L}_a : G \longrightarrow G$ is an isometry of h. This means that, for any $a, b \in G$ and any $u, v \in T_bG$,

$$h\left(T_e\mathcal{L}_a(u), T_e\mathcal{L}_a(v)\right) = h(u, v).$$

We call Riemannian Lie group a Lie group with a left invariant Riemannian metric.

Proposition 1.3.1 *Let h be a Riemannian metric on G. Then the following assertions are equivalent.*

- (i) *The metric h is left invariant.*
- (ii) *For any couple of left invariant vector fields X, Y, $h(X, Y)$ is a constant function.*

Proof. If $a, b \in G$ and $u, v \in T_bG$. There exists an unique couple of left invariant vector fields Y, Z such that $Y(b) = u$ and $Z(b) = v$. Then

$$h\left(T_e\mathcal{L}_a(u), T_e\mathcal{L}_a(v)\right) = h\left(T_e\mathcal{L}_a(Y(b)), T_e\mathcal{L}_a(Z(b))\right) = h(Y(ab), Z(ab)).$$

This shows that (i) is equivalent to (ii) and completes the proof. □

Proposition 1.3.2 *Let h be a Riemannian metric on a connected Lie group G. Then the following assertions are equivalent.*

(*i*) *The metric h is left invariant.*

(*ii*) *The Lie algebra of right vector fields $\mathcal{X}^r(G)$ is a Lie subalgebra of the Lie algebra of Killing vector fields* $\mathrm{Kill}(h)$.

Proof. If X is a right invariant vector field then it is complete and its flow ϕ^X is given, for any $t \in \mathbb{R}$, $\phi_t^X = \mathcal{L}_{\exp(tX(e))}$. This shows that (i) implies (ii).
On the other hand, let X be a right invariant vector field and Y and Z a couple of left invariant vector field. Since $[X, Y] = [X, Z] = 0$, we get

$$L_X h(Y, Z) = X.h(Y, Z).$$

This shows that (ii) implies (i) of Proposition 1.3.1 and hence (i). □

Let G be a Lie group. We denote by $\mathcal{M}^\ell(G)$ the set of left invariant Riemannian metrics on G and $\mathcal{M}(\mathfrak{g})$ the set of definite positive inner products on $\mathfrak{g}$. According to Proposition 1.3.1, the map $\mathcal{I} : \mathcal{M}^\ell(G) \longrightarrow \mathcal{M}(\mathfrak{g})$, $h \mapsto h(e)$ is a bijection. Moreover, we have:

Proposition 1.3.3 *Let (G, h) be a Riemannian Lie group. Then the tensor curvature R, the sectional curvature Q, the Ricci curvature* ric *and the scalar curvature $\mathfrak{s}$ of (G, h) are left invariant. Thus for any left invariant vector field X, Y, Z and for any $a \in G$,*

$$R(X(a), Y(a))Z(a) = T_e\mathcal{L}_a(R(X(e), Y(e))Z(e)),\ \mathrm{ric}(X(a), Y(a)) = \mathrm{ric}(X(e), Y(e)).$$

Moreover, $\mathfrak{s}$ is a constant and $Q(X(a), Y(a)) = Q(X(e), Y(e))$.

Proof. This an immediate consequence of the fact that the metric is left invariant. □

This proposition shows that the sign of the sectional curvature, Ricci curvature or scalar curvature of a Riemannian Lie group is entirely determined by the values of these curvatures on the neutral element. We will devote the next section to give useful formulas of the different curvatures on the Lie algebra of a Riemannian Lie group.

To end this section, we point out that a left invariant Riemannian metric on a Lie group is geodesically complete.

1.3.2 Curvatures on the Lie algebra of a Riemannian Lie group

Let (G, h) be a Riemannian Lie algebra, $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ its Lie algebra and $\langle\ ,\ \rangle = h(e)$. For any $u \in \mathfrak{g}$, we denote by $\mathrm{ad}_u : \mathfrak{g} \longrightarrow \mathfrak{g}$ the endomorphism given by $\mathrm{ad}_u v = [u, v]_{\mathfrak{g}}$ and for any endomorphism $F : \mathfrak{g} \longrightarrow \mathfrak{g}$, we denote by $F^* : \mathfrak{g} \longrightarrow \mathfrak{g}$ the adjoint of F with respect to $\langle\ ,\ \rangle$ given by

$$\langle F(u), v\rangle = \langle u, F^*(v)\rangle,\ \forall u, v \in \mathfrak{g}.$$

Through this paper, we denote by $Z(\mathfrak{g})$ the centre of $\mathfrak{g}$ and $\mathfrak{D}(\mathfrak{g})$ its derived ideal. We denote also by $B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ the Killing form given by

$$B(u, v) = \mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v).$$

We call a Lie group G unimodular if for any $u \in \mathfrak{g}$, $\mathrm{tr}(\mathrm{ad}_u) = 0$.[1]

We have shown in Proposition 1.3.2 that any right invariant vector field is a Killing vector field for (G, h), however, not any left invariant vector field is a Killing vector field. So we consider

$$\mathfrak{L}(\mathfrak{g}) = \{u \in \mathfrak{g},\ u^\ell \in \mathrm{Kill}(h)\}.$$

It is clear that $\mathfrak{L}(\mathfrak{g})$ is subalgebra of $\mathfrak{g}$, and from (1.14) we can deduce that

$$\mathfrak{L}(\mathfrak{g}) = \{u \in \mathfrak{g},\ \mathrm{ad}_u + \mathrm{ad}_u^* = 0\}. \tag{1.16}$$

Note that the centre $Z(\mathfrak{g})$ of $\mathfrak{g}$ is contained in $\mathfrak{L}(\mathfrak{g})$.

The Levi-Civita product on $\mathfrak{g}$ is the product $\mathrm{L} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, $(u, v) \mapsto \mathrm{L}_u v$

[1]One can see [13] for a precise definition of unimodular Lie groups.

given by

$$\mathrm{L}_u v = (\nabla_{u^\ell} v^\ell)(e),$$

where ∇ is the Levi-Civita connection associated to (G,h). By using (1.4), we get for any $u,v,w \in \mathfrak{g}$

$$\langle \mathrm{L}_u v, w\rangle = \frac{1}{2}\left\{\langle [u,v]_{\mathfrak{g}}, w\rangle + \langle [w,u]_{\mathfrak{g}}, v\rangle + \langle [w,v]_{\mathfrak{g}}, u\rangle\right\}. \tag{1.17}$$

For any $u \in \mathfrak{g}$, we denote by $\mathrm{R}_u, \mathrm{J}_u, \mathrm{ad}_u : \mathfrak{g} \longrightarrow \mathfrak{g}$ the endomorphisms of $\mathfrak{g}$ given by $\mathrm{R}_u v = \mathrm{L}_v u$ and $\mathrm{J}_u v = \mathrm{ad}_v^* u$. It is easy to check that J_u is a skew-symmetric endomorphism and $\mathrm{J}_u = 0$ if and only if $u \in \mathfrak{D}(\mathfrak{g})^\perp$.

Proposition 1.3.4 *We have the following formulas:*

(*i*) *For any* $u,v \in \mathfrak{g}$, $\mathrm{L}_u v - \mathrm{L}_v u = [u,v]_{\mathfrak{g}}$, *i.e.*, $\mathrm{L}_u - \mathrm{R}_u = \mathrm{ad}_u$.

(*ii*) *For any* $u,v,w \in \mathfrak{g}$, $\langle \mathrm{L}_u v, w\rangle + \langle v, \mathrm{L}_u w\rangle = 0$. *This means that , for any* $u \in \mathfrak{g}$, L_u *is skew-symmetric, i.e.*, $\mathrm{L}_u^* = -\mathrm{L}_u$.

(*iii*) *For any* $u \in \mathfrak{g}$, $\mathrm{L}_u = \frac{1}{2}(\mathrm{ad}_u - \mathrm{ad}_u^*) - \frac{1}{2}\mathrm{J}_u$.

(*iv*) *For any* $u \in \mathfrak{g}$, $\mathrm{R}_u = -\frac{1}{2}(\mathrm{ad}_u + \mathrm{ad}_u^*) - \frac{1}{2}\mathrm{J}_u$.

Proof. The assertions (*iii*) and (*iv*) follow easily from (1.17), (*i*) and (*ii*) follow from (*iii*) and (*iv*). □

Remark 1 *One can deduce from Proposition 1.3.4* (*i*) *that*

$$\mathfrak{L}(\mathfrak{g}) = \{u \in \mathfrak{g},\ \mathrm{R}_u + \mathrm{R}_u^* = 0\}.$$

We denote by $\mathrm{K} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ the curvature of (G,h) at e. We have, for any $u,v \in \mathfrak{g}$,

$$\mathrm{K}(u,v) = \mathrm{L}_{[u,v]_{\mathfrak{g}}} - [\mathrm{L}_u, \mathrm{L}_v].$$

According to Proposition 1.2.4, K satisfies:

1. $\mathrm{K}(u,v) = -\mathrm{K}(v,u)$ (skew-symmetry);

2. $\mathrm{K}(u,v)w+\mathrm{K}(v,w)u+\mathrm{K}(w,u)v=0$ (Bianchi's identity);

3. $\langle \mathrm{K}(u,v)w,z\rangle = -\langle \mathrm{K}(u,v)z,w\rangle$;

4. $\langle \mathrm{K}(u,v)z,w\rangle = \langle \mathrm{K}(z,w)u,v\rangle$.

Proposition 1.3.5 *For any* $u,v\in\mathfrak{g}$,

$$\langle \mathrm{K}(u,v)u,v\rangle \quad = \quad -\frac{3}{4}|\mathrm{ad}_u v|^2+\frac{1}{4}|\mathrm{ad}_u^* v+\mathrm{ad}_v^* u|^2-\frac{1}{2}\langle \mathrm{ad}_u v,\mathrm{ad}_u^* v-\mathrm{ad}_v^* u\rangle-\langle \mathrm{ad}_u^* u,\mathrm{ad}_v^* v\rangle.$$

If the sectional curvature is negative (resp. strictly negative) then $\mathfrak{L}(\mathfrak{g})\subset\mathfrak{D}(\mathfrak{g})^{\perp}$ *(resp.* $\mathfrak{L}(\mathfrak{g})=\{0\}$*).*

Proof. The formula follows from a direct computation using (1.17) and (*i*) of Proposition 1.3.4. Indeed,

$$\begin{aligned}\langle \mathrm{K}(u,v)u,v\rangle &= \langle \mathrm{L}_{[u,v]_\mathfrak{g}}u,v\rangle-\langle \mathrm{L}_u\mathrm{L}_v u,v\rangle+\langle \mathrm{L}_v\mathrm{L}_u u,v\rangle\\ &= \frac{1}{2}\langle[[u,v]_\mathfrak{g},u]_\mathfrak{g},v\rangle+\frac{1}{2}\langle[v,[u,v]_\mathfrak{g}]_\mathfrak{g},u\rangle+\frac{1}{2}\langle[v,u]_\mathfrak{g},[u,v]_\mathfrak{g}\rangle\\ &\quad+\langle \mathrm{L}_v u,\mathrm{L}_u v\rangle-\langle \mathrm{L}_u u,\mathrm{L}_v v\rangle\\ &= \frac{1}{2}\langle[[u,v]_\mathfrak{g},u]_\mathfrak{g},v\rangle+\frac{1}{2}\langle[v,[u,v]_\mathfrak{g}]_\mathfrak{g},u\rangle+\frac{1}{2}\langle[v,u]_\mathfrak{g},[u,v]_\mathfrak{g}\rangle\\ &\quad+\langle[v,u]_\mathfrak{g},\mathrm{L}_u v\rangle+\langle \mathrm{L}_u v,\mathrm{L}_u v\rangle-\langle \mathrm{L}_u u,\mathrm{L}_v v\rangle\\ &= \frac{1}{2}\langle[[u,v]_\mathfrak{g},u]_\mathfrak{g},v\rangle+\frac{1}{2}\langle[v,[u,v]_\mathfrak{g}]_\mathfrak{g},u\rangle+\frac{1}{2}\langle[v,u]_\mathfrak{g},[u,v]_\mathfrak{g}\rangle\\ &\quad-\frac{1}{2}|[u,v]_\mathfrak{g}|^2+\frac{1}{2}\langle[[v,u]_\mathfrak{g},u]_\mathfrak{g},v\rangle+\frac{1}{2}\langle[[v,u]_\mathfrak{g},v]_\mathfrak{g},u\rangle\\ &\quad+\langle \mathrm{L}_u v,\mathrm{L}_u v\rangle-\langle \mathrm{L}_u u,\mathrm{L}_v v\rangle\\ &= |\mathrm{L}_u v|^2-\langle \mathrm{L}_u u,\mathrm{L}_v v\rangle-\langle \mathrm{ad}_v\circ\mathrm{ad}_v u,u\rangle-|[u,v]_\mathfrak{g}|^2.\end{aligned}$$

The desired formula follows from the relation (see Proposition 1.3.4 (*iii*))

$$\mathrm{L}_u v=\frac{1}{2}(\mathrm{ad}_u v-\mathrm{ad}_u^* v-\mathrm{ad}_v^* u).$$

Indeed, we have

$$\begin{aligned}
\langle \mathrm{K}(u,v)u,v\rangle &= |\mathrm{L}_u v|^2 - \langle \mathrm{L}_u u, \mathrm{L}_v v\rangle - \langle \mathrm{ad}_v\circ \mathrm{ad}_v u, u\rangle - |[u,v]_{\mathfrak{g}}|^2\\
&= \frac{1}{4}|\mathrm{ad}_u v - \mathrm{ad}_u^* v - \mathrm{ad}_v^* u|^2 - \langle \mathrm{ad}_u^* u, \mathrm{ad}_v^* v\rangle - \langle \mathrm{ad}_v u, \mathrm{ad}_v^* u\rangle - |\mathrm{ad}_u v|^2\\
&= \frac{1}{4}|\mathrm{ad}_u v|^2 + \frac{1}{4}|\mathrm{ad}_u^* v|^2 + \frac{1}{4}|\mathrm{ad}_v^* u|^2 - \frac{1}{2}\langle \mathrm{ad}_u v, \mathrm{ad}_u^* v\rangle - \frac{1}{2}\langle \mathrm{ad}_u v, \mathrm{ad}_v^* u\rangle\\
&\quad + \frac{1}{2}\langle \mathrm{ad}_u^* v, \mathrm{ad}_v^* u\rangle - \langle \mathrm{ad}_u^* u, \mathrm{ad}_v^* v\rangle - \langle \mathrm{ad}_v u, \mathrm{ad}_v^* u\rangle - |\mathrm{ad}_u v|^2\\
&= -\frac{3}{4}|\mathrm{ad}_u v|^2 + \frac{1}{4}|\mathrm{ad}_u^* v|^2 + \frac{1}{4}|\mathrm{ad}_v^* u|^2 - \frac{1}{2}\langle \mathrm{ad}_u v, \mathrm{ad}_u^* v\rangle + \frac{1}{2}\langle \mathrm{ad}_u v, \mathrm{ad}_v^* u\rangle\\
&\quad + \frac{1}{2}\langle \mathrm{ad}_u^* v, \mathrm{ad}_v^* u\rangle - \langle \mathrm{ad}_u^* u, \mathrm{ad}_v^* v\rangle\\
&= -\frac{3}{4}|\mathrm{ad}_u v|^2 + \frac{1}{4}|\mathrm{ad}_u^* v + \mathrm{ad}_v^* u|^2 - \frac{1}{2}\langle \mathrm{ad}_u v, \mathrm{ad}_u^* v - \mathrm{ad}_v^* u\rangle - \langle \mathrm{ad}_u^* u, \mathrm{ad}_v^* v\rangle.
\end{aligned}$$

Let $u \in \mathfrak{L}(\mathfrak{g})$. By using $\mathrm{ad}_u + \mathrm{ad}_u^* = 0$ and the formula we have just established, we get for any $v \in \mathfrak{g}$,

$$\langle \mathrm{K}(u,v)u,v\rangle = \frac{1}{4}|\mathrm{ad}_v^* u|^2 \geq 0.$$

If the sectional curvature is negative then $\mathrm{ad}_v^* u = 0$ and hence $u \in \mathfrak{D}(\mathfrak{g})^{\perp}$.
If the sectional curvature is strictly negative then $u = 0$. □

Example 2 *Let G be a non abelian n-dimensional Lie group such that its Lie algebras satisfies the following properties: there exists a linear form $\ell : \mathfrak{g} \longrightarrow \mathbb{R}$ such that, for any $u, v \in \mathfrak{g}$,*

$$[u,v]_{\mathfrak{g}} = \ell(u)v - \ell(v)u.$$

Let h be a left invariant metric on G and $\langle\ ,\ \rangle$ its value on e. Let compute the curvature of h at e by using Proposition 1.3.5. Let u, v be two vector such that $\langle u, v\rangle = 0$, $|u| = |v| = 1$. We complete to have an orthonormal basis $(u, v, e_1, \ldots, e_{n-2})$. A direct computation shows:

$$\mathrm{ad}_u^* v = \ell(u)v,\ \mathrm{ad}_v^* u = \ell(v)u,\ \mathrm{ad}_u^* u = -\ell(v)v - \sum_{i=1}^{n-2}\ell(e_i)e_i,\ \mathrm{ad}_v^* v = -\ell(u)u - \sum_{i=1}^{n-2}\ell(e_i)e_i.$$

By using Proposition 1.3.5, we get

$$Q(u,v) = -\ell(u)^2 - \ell(v)^2 - \sum_{i=1}^{n-2} \ell(e_i)^2 = -|\ell|^2 < 0.$$

So the sectional curvature is a strictly negative constant.

Recall that the Ricci curvature at e is defined by $\mathrm{ric}(u,v) = \mathrm{tr}\,(w \longrightarrow \mathrm{K}(u,w)v)$. In order to give an useful formula of ric we introduce the *mean curvature vector* on $\mathfrak{g}$ which is the vector given by

$$\langle H, u\rangle = \mathrm{tr}(\mathrm{ad_u}),\ \forall u \in \mathfrak{g}. \tag{1.18}$$

It follows that G is unimodular if and only if $H = 0$. Since $\mathrm{tr}(\mathrm{ad}_{[u,v]_\mathfrak{g}}) = \mathrm{tr}([\mathrm{ad}_u, \mathrm{ad}_v]) = 0$ then $H \in \mathfrak{D}(\mathfrak{g})^\perp$.

Proposition 1.3.6 *1. If $\tau(u,v) : \mathfrak{g} \longrightarrow \mathfrak{g}$ is the endomorphism given by $\tau(u,v)w = \mathrm{K}(u,w)v$ then*

$$\tau(u,v) = -\mathrm{R}_v \circ \mathrm{R}_u + \mathrm{R}_{u.v} - [\mathrm{L}_u, \mathrm{R}_v],$$

where $u.v = \mathrm{L}_u v$.

2. For any $u, v \in \mathfrak{g}$,

$$\mathrm{ric}(u,v) = -\frac{1}{2}B(u,v) - \frac{1}{2}\mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v^*) - \frac{1}{4}\mathrm{tr}(\mathrm{J}_u \circ \mathrm{J}_v) - \frac{1}{2}(\langle \mathrm{ad}_H u, v\rangle + \langle \mathrm{ad}_H v, u\rangle).$$

Proof.

1. For any $u, v, w \in \mathfrak{g}$,

$$\begin{aligned}
\tau(u,v)w &= \mathrm{K}(u,w)v \\
&= \mathrm{L}_{[u,w]_\mathfrak{g}} v - \mathrm{L}_u \circ \mathrm{L}_w v + \mathrm{L}_w \circ \mathrm{L}_u v \\
&= \mathrm{R}_v \circ \mathrm{ad}_u w - \mathrm{L}_u \circ \mathrm{R}_v w + \mathrm{R}_{u.v} w \\
&= \mathrm{R}_v \circ \mathrm{L}_u w - \mathrm{R}_v \circ \mathrm{R}_u w - \mathrm{L}_u \circ \mathrm{R}_v w + \mathrm{R}_{u.v} w \quad \text{(Proposition 1.3.4 } (i)) \\
&= -\mathrm{R}_v \circ \mathrm{R}_u w + \mathrm{R}_{u.v} w + [\mathrm{R}_v, \mathrm{L}_u] w.
\end{aligned}$$

This establishes the first formula.

2. From the first formula, we get

$$\mathrm{ric}(u,v)=\mathrm{tr}(\tau(u,v))=-\mathrm{tr}\left(\mathrm{R}_v\circ\mathrm{R}_u\right)+\mathrm{tr}(\mathrm{R}_{u.v}).$$

Choose an orthonormal basis $(e_i)_{i=1}^n$ of $\mathfrak{g}$. We have

$$\begin{aligned}\mathrm{tr}(\mathrm{R}_{u.v}) &= \sum_{i=1}^n\langle \mathrm{R}_{u.v}e_i,e_i\rangle\\ &= \sum_{i=1}^n\langle \mathrm{L}_{e_i}\circ\mathrm{L}_u v,e_i\rangle\\ &\overset{(1.17)}{=} -\sum_{i=1}^n\langle[\mathrm{L}_u v,e_i]_{\mathfrak{g}},e_i\rangle\\ &= -\mathrm{tr}(\mathrm{ad}_{u.v})\\ &\overset{(1.18)}{=} -\langle H,\mathrm{L}_u v\rangle\\ &\overset{(1.17)}{=} -\frac12\langle \mathrm{ad}_H u,v\rangle-\frac12\langle \mathrm{ad}_H v,u\rangle-\frac12\mathrm{tr}(\mathrm{ad}_{[u,v]_{\mathfrak{g}}})\\ &= -\frac12\langle \mathrm{ad}_H u,v\rangle-\frac12\langle \mathrm{ad}_H v,u\rangle.\end{aligned}$$

So

$$\mathrm{ric}(u,v)=-\mathrm{tr}\left(\mathrm{R}_v\circ\mathrm{R}_u\right)-\frac12\langle \mathrm{ad}_H u,v\rangle-\frac12\langle \mathrm{ad}_H v,u\rangle. \tag{1.19}$$

To achieve the proof, from Proposition 1.3.4 (*iv*), we have

$$\mathrm{R}_u=-\frac12(\mathrm{ad}_u+\mathrm{ad}_u^*)-\frac12\mathrm{J}_u\quad\text{and}\quad \mathrm{R}_v=-\frac12(\mathrm{ad}_v+\mathrm{ad}_v^*)-\frac12\mathrm{J}_v.$$

By replacing in (1.19) and using the fact that, since $\mathrm{ad}_u+\mathrm{ad}_u^*$ is symmetric and J_v skew-symmetric then $\mathrm{tr}\left((\mathrm{ad}_u+\mathrm{ad}_u^*)\circ\mathrm{J}_v\right)=0$, we get the desired formula.
□

Corollary 1.3.1 *Let (G,h) be a Riemannian nilpotent Lie group. Then its Ricci curvature at e is given by*

$$\mathrm{ric}(u,v)=-\frac12\mathrm{tr}(\mathrm{ad}_u\circ\mathrm{ad}_v^*)-\frac14\mathrm{tr}(\mathrm{J}_u\circ\mathrm{J}_v),$$

for any $u, v \in \mathfrak{g}$.

Proof. It is a consequence of Proposition 1.3.6 and the fact that the Lie algebra of G is nilpotent and hence, for any $u, v \in \mathfrak{g}$, $\mathrm{tr}(\mathrm{ad}_u) = \mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v) = 0$, in particular $H = 0$. $\square$

Define now the auto-adjoint endomorphisms Ric, $\widehat{B}$, $\mathcal{J}_1$ and $\mathcal{J}_2$ by

$$\begin{aligned}\langle \mathrm{Ric}\, u, v\rangle &= \mathrm{ric}(u, v),\\ \langle \widehat{B}u, v\rangle &= B(u, v),\\ \langle \mathcal{J}_1 u, v\rangle &= \mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v^*),\\ \langle \mathcal{J}_2 u, v\rangle &= -\mathrm{tr}(\mathrm{J}_u \circ \mathrm{J}_v).\end{aligned}$$

From Proposition 1.3.6, we get

$$\mathrm{Ric} = -\frac{1}{2}\left(\widehat{B} + \mathcal{J}_1\right) + \frac{1}{4}\mathcal{J}_2 - \frac{1}{2}(\mathrm{ad}_H + \mathrm{ad}_H^*). \tag{1.20}$$

Proposition 1.3.7 *With the notations above we have:*

(i) $\mathcal{J}_1$, $\mathcal{J}_2$ *and* $\widehat{B} + \mathcal{J}_1$ *are positive skew-symmetric endomorphisms and hence their eigenvalues are real positive.*

(ii) $\ker \mathcal{J}_1 = Z(\mathfrak{g})$, $\ker \mathcal{J}_2 = \mathfrak{D}(\mathfrak{g})^\perp$ *and* $\ker(\widehat{B} + \mathcal{J}_1) = \mathfrak{L}(\mathfrak{g})$.

(iii) For any orthonormal basis $(e_1, \ldots, e_n)$ *of* $\mathfrak{g}$,

$$\mathcal{J}_1 = \sum_{i=1}^n \mathrm{ad}_{e_i}^* \circ \mathrm{ad}_{e_i} \quad \text{and} \quad \mathcal{J}_2 = \sum_{i=1}^n \mathrm{ad}_{e_i} \circ \mathrm{ad}_{e_i}^*,$$

in particular $\mathrm{tr}\mathcal{J}_1 = \mathrm{tr}\mathcal{J}_2 \geq 0$ *and* $\mathrm{tr}\mathcal{J}_1 = 0$ *if and only if* $\mathfrak{g}$ *is abelian.*

Proof.

(i) This is a consequence of the following relations

$$\langle \mathcal{J}_1 u, u\rangle = \mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_u^*) \geq 0,\ \langle \mathcal{J}_2 u, u\rangle = \mathrm{tr}(\mathrm{J}_u \circ \mathrm{J}_u^*) \geq 0,$$

and

$$\langle (\widehat{B} + \mathcal{J}_1)u, u\rangle = \frac{1}{2}\mathrm{tr}((\mathrm{ad}_u + \mathrm{ad}_u^*)^2) \geq 0.$$

(*ii*) This is a consequence of the relations above and the fact that $\mathrm{J}_u = 0$ if and only if $u \in \mathfrak{D}(\mathfrak{g})^\perp$.

(*iii*) We have

$$\begin{aligned} \langle \mathcal{J}_1 u, v\rangle &= \mathrm{tr}(\mathrm{ad}_u^* \circ \mathrm{ad}_v) \\ &= \sum_{i=1}^n \langle \mathrm{ad}_u^* \circ \mathrm{ad}_v e_i, e_i\rangle \\ &= \sum_{i=1}^n \langle [v, e_i], [u, e_i]_{\mathfrak{g}}\rangle \\ &= \sum_{i=1}^n \langle \mathrm{ad}_{e_i}^* \circ \mathrm{ad}_{e_i} u, v\rangle. \end{aligned}$$

This shows the first relation, the second one can be obtained in the same way.
□

From (1.20) and the relation $\mathrm{tr}\mathcal{J}_1 = \mathrm{tr}\mathcal{J}_2$, we deduce that the scalar curvature of $\langle\ ,\ \rangle$ is given by

$$\mathfrak{s} = -\frac{1}{4}\left(2\mathrm{tr}(\widehat{B}) + \mathrm{tr}\mathcal{J}_1\right) - |H|^2. \tag{1.21}$$

Since for a nilpotent Lie algebra $B = 0$ and $H = 0$ we deduce:

Proposition 1.3.8 *Let G be a Riemannian nilpotent non abelian Lie group. Then its scalar curvature is given by $\mathfrak{s} = -\frac{1}{4}\mathrm{tr}\mathcal{J}_1 < 0$.*

1.3.3 Curvature of bi-invariant Riemannian metrics on Lie groups

A Riemannian metric on a Lie group is bi-invariant if it is both left and right invariant.

Proposition 1.3.9 *Let G be a Lie group and h is a left invariant Riemannian metric on G. If h is also right invariant then $\mathfrak{L}(\mathfrak{g}) = \mathfrak{g}$, i.e., for any $u \in \mathfrak{g}$, $\mathrm{ad}_u + \mathrm{ad}_u^* = 0$. If G is connected the converse is also true.*

Proof. We have seen in Proposition 1.3.1 that if h is left invariant then $\mathcal{X}^r(G) \subset \mathrm{Kill}(h)$. In a similar way if h is right invariant then $\mathcal{X}^\ell(G) \subset \mathrm{Kill}(h)$ which is equivalent to $\mathfrak{L}(\mathfrak{g}) = \mathfrak{g}$. As in Proposition 1.3.1 the converse is true if G is connected. □

Proposition 1.3.10 *Let (G, h) be a Lie group with a bi-invariant Riemannian metric. Then, for any $u, v \in \mathfrak{g}$,*

$$\mathrm{L}_u = \frac{1}{2}\mathrm{ad}_u,\ \mathrm{K}(u,v) = \frac{1}{4}\mathrm{ad}_{[u,v]},\ \langle \mathrm{K}(u,v)u, v\rangle = \frac{1}{4}|[u,v]|^2,\ \mathrm{ric}(u,v) = -\frac{1}{4}B(u,v).$$

In particular, the sectional curvature and the Ricci curvature of h are positive and $\mathrm{ric}(u,u) = 0$ *if and only if $u \in Z(\mathfrak{g})$.*

Proof. It is a an immediate consequence of the fact that $\mathrm{ad}_u^* = -\mathrm{ad}_u$, $\mathrm{J}_u = \mathrm{ad}_u$, $H = 0$ and Propositions 1.3.4 and 1.3.6. Moreover, for any $u \in \mathfrak{g}$, $\mathrm{ric}(u,u) = \frac{1}{4}\mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_u^*) \geq 0$ and $\mathrm{ric}(u,u) = 0$ if and only if $\mathrm{ad}_u = 0$. □

Theorem 1.3.1 *(i) A connected Lie group carries a bi-invariant Riemannian metric if and only if it is isomorphic to the cartesian product of a compact Lie group and an abelian Lie group.*

(ii) If the Lie algebra of a compact Lie group G is simple then there is up to multiplication by a strictly positive constant an unique bi-invariant Riemannian metric and this metric is Einstein.

Proof.

(*i*) Let $(\mathfrak{g}, [\ ,\ ]_\mathfrak{g}, \langle\ ,\ \rangle)$ be the Lie algebra of a Riemannian Lie group (G, h) endowed with a bi-invariant Riemannian metric. From the relation

$$\langle [u,v]_\mathfrak{g}, w\rangle + \langle v, [u,w]_\mathfrak{g}\rangle,$$

we can deduce easily that $\mathfrak{g}$ splits orthogonally

$$\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{D}(\mathfrak{g}).$$

The Killing form of $\mathfrak{D}(\mathfrak{g})$ is given by

$$B(u,v) = \mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v) = -\mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v^*),$$

from any $u, v \in \mathfrak{D}(\mathfrak{g})$. This formula shows that B is definite negative and hence $\mathfrak{D}(\mathfrak{g})$ is semi-simple compact (see for instance [8]). So the universal covering $\widetilde{G}$ of G splits as the cartesian product of a compact Lie H group and an abelian group $\mathbb{R}^m$ and $G = \widetilde{G}/\Gamma$ where Γ is a discrete normal subgroup of $\widetilde{G}$. Projecting Γ into $\mathbb{R}^m$, let V be the vector space spanned by its image and let $V^\perp$ be the orthogonal complement in $\mathbb{R}^m$. Then G is the cartesian product of the compact Lie group $(H \times V)/\Gamma$ and the vector group $V^\perp$.

Conversely, any left invariant metric on an abelian Lie group is also bi-invariant and let G be a compact Lie group. Choose an arbitrary metric μ on $\mathfrak{g}$ and define $\langle\ ,\ \rangle$ on $\mathfrak{g}$ by

$$\langle u, v\rangle = \int_G \mu(\mathrm{Ad}_g u, \mathrm{Ad}_g v) d\lambda,$$

where λ is a Haar measure on G. Then the left invariant Riemannian metric on G associated to $\langle\ ,\ \rangle$ is clearly also right invariant.

(ii) Suppose that the Lie algebra of G is simple, pick two bi-invariant Riemannian metrics h_1 and h_2 on G and denote by $\langle\ ,\ \rangle_1$, $\langle\ ,\ \rangle_2$ their value at e. For any $u, v \in \mathfrak{g}$, the relation

$$\langle u, v\rangle_2 = \langle Au, v\rangle_1 = \langle u, Av\rangle_1$$

defines a symmetric (with respect both $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$) isomorphism of $\mathfrak{g}$. Since h_1 and h_2 are bi-invariant and according to Proposition 1.3.9, we have for any $u, v, w \in \mathfrak{g}$,

$$\begin{aligned}
\langle [u,v]_{\mathfrak{g}}, w\rangle_2 &= \langle [u,v]_{\mathfrak{g}}, Aw\rangle_1 \\
&= -\langle [u,Aw]_{\mathfrak{g}}, v\rangle_1 \\
&= -\langle [u,Aw]_{\mathfrak{g}}, A^{-1}v\rangle_2 \\
&= \langle Aw, [u,A^{-1}v]_{\mathfrak{g}}\rangle_2 \\
&= \langle w, A[u,A^{-1}v]_{\mathfrak{g}}\rangle_2.
\end{aligned}$$

This relation shows that, for any $u \in \mathfrak{g}$, $\mathrm{ad}_u \circ A = A \circ \mathrm{ad}_u$. This implies that, for any real eigenvalue $\lambda \in \mathfrak{g}$ of A, the associated eigenspace is an ideal of $\mathfrak{g}$. Now $\mathfrak{g}$ is simple and A is diagonalizable over $\mathbb{R}$ so we must have $A = \lambda \mathrm{Id}_{\mathfrak{g}}$ and hence $\langle\ ,\ \rangle_2 = \lambda\langle\ ,\ \rangle_1$ and finally $h_2 = \lambda h_1$.
The Ricci curvature of a bi-invariant metric $\langle\ ,\ \rangle$ on $\mathfrak{g}$ satisfies, according to Proposition 1.3.10, $\mathrm{ric} = -\frac{1}{4}B$ where B is the Killing form of $\mathfrak{g}$. Since the Killing form is bi-invariant argument similar to what above gives $B = \mu\langle\ ,\ \rangle$ which shows that $\langle\ ,\ \rangle$ is Einstein. □

Corollary 1.3.2 *Every compact Lie group admits a left invariant (and in fact bi-invariant) metric so that all sectional curvatures are positive.*

1.4 Sectional curvature of Riemannian Lie groups

1.4.1 Riemannian Lie groups with strictly positive sectional curvature

The following theorem have been proved in [14], we give it without proof since it needs the introduction of tools which are beyond this course.

Theorem 1.4.1 *The 3-sphere group* $\mathrm{SU}(2)$*, consisting of* 2×2 *unitary matrices of determinant 1, is the only simply connected Lie group which admits a left invariant metric of strictly positive sectional curvature.*

1.4.2 Flat Riemannian Lie groups

Let (G, h) be a Riemannian Lie group and $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \langle\ ,\ \rangle)$ its Lie algebra endowed with $\langle\ ,\ \rangle = h(e)$. One can see easily that the orthogonal of the derived ideal of $\mathfrak{g}$ is given by

$$\mathfrak{D}(\mathfrak{g})^{\perp} = \{u \in \mathfrak{g}, \mathrm{R}_u = \mathrm{R}_u^*\}. \tag{1.22}$$

Put

$$N_\ell(\mathfrak{g}) = \{u \in \mathfrak{g}, \mathrm{L}_u = 0\} \quad \text{and} \quad N_r(\mathfrak{g}) = \{u \in \mathfrak{g}, \mathrm{R}_u = 0\}\,.$$

We have obviously

$$N_r(\mathfrak{g}) = (\mathfrak{g}\mathfrak{g})^{\perp}, \tag{1.23}$$

where $\mathfrak{g}\mathfrak{g} = \mathrm{span}\{\mathrm{L}_u v,\ u, v \in \mathfrak{g}\}$.

Proposition 1.4.1 *If the curvature of h vanishes then $H = 0$ and*

$$\mathfrak{L}(\mathfrak{g}) = \mathfrak{D}(\mathfrak{g})^{\perp} = \{u \in \mathfrak{g},\ \mathrm{R}_u = 0\}.$$

Proof. By using (1.17) one can see easily that, for any $u \in \mathfrak{L}(\mathfrak{g}) \cup \mathfrak{D}(\mathfrak{g})^{\perp}$, $\mathrm{L}_u u = 0$ and deduce from Proposition 1.3.6 (i) that

$$[\mathrm{R}_u, \mathrm{L}_u] = \mathrm{R}_u^2.$$

From this relation, one can deduce by induction that for any $k \in \mathbb{N}^*$

$$[\mathrm{R}_u^k, \mathrm{L}_u] = k\mathrm{R}_u^{k+1},$$

and hence $\mathrm{tr}(\mathrm{R}_u^k) = 0$ for any $k \geq 2$ which implies that R_u is nilpotent. A skew-symmetric or a symmetric endomorphism is nilpotent iff it vanishes so

$$\mathfrak{L}(\mathfrak{g}) = \mathfrak{D}(\mathfrak{g})^{\perp} = \{u \in \mathfrak{g},\ \mathrm{R}_u = 0\}.$$

On the other hand, for any $u, v \in \mathfrak{g}$, $\mathrm{tr}(\mathrm{ad}_{[u,v]}) = 0$, thus $H \in \mathfrak{D}(\mathfrak{g})^{\perp}$. Now, for any $u \in \mathfrak{D}(\mathfrak{g})^{\perp}$, $\mathrm{R}_u = 0$ and hence

$$\mathrm{tr}(\mathrm{ad}_u) = \mathrm{tr}(\mathrm{R}_u) = \langle H, u\rangle = 0,$$

which implies $H \in \mathfrak{D}(\mathfrak{g})$ and hence $H = 0$. $\square$

A Riemannian Lie group (G,h) is flat if its curvature at e vanishes identically. This is equivalent to the fact that $\mathfrak{g}$ endowed with the Levi-Civita product is a left symmetric algebra, i.e., for any $u,v,w\in\mathfrak{g}$,

$$\mathrm{ass}(u,v,w)=\mathrm{ass}(v,u,w),$$

where $\mathrm{ass}(u,v,w)=(u.v).w-u.(v.w)$ and $u.v=\mathrm{L}_u v$. Left symmetric algebras (or under other names like pre-algebras, quasi-associative algebras, Vinberg algebras) constitute an important topic in both algebra and geometry. For a left invariant connection on a Lie group its flatness is equivalent to to left symmetry of the induced product on the Lie algebra. The following theorem was proved first in [13] using some strong geometric theorems. We give here a more precise statement and an algebraic proof based on an important result about left symmetric Lie algebras, namely the fact proved in [9] that for a left symmetric algebra $\mathfrak{g}$, $\mathfrak{g}\neq\mathfrak{D}(\mathfrak{g})$. The proof we will give appeared first in [1].

Theorem 1.4.2 *A Riemannian Lie group Lie group (G,h) is flat if and only $\mathfrak{L}(\mathfrak{g})$ and $\mathfrak{D}(\mathfrak{g})$ are abelian, $\mathfrak{D}(\mathfrak{g})^\perp=\mathfrak{L}(\mathfrak{g})$ and*

$$\mathfrak{g}=\mathfrak{L}(\mathfrak{g})\oplus\mathfrak{D}(\mathfrak{g}).$$

Moreover, in this case the dimension of $\mathfrak{D}(\mathfrak{g})$ is even and the Levi-Civita product is given by

$$\mathrm{L}_a=\begin{cases}\mathrm{ad}_a & \text{if}\quad a\in\mathfrak{L}(\mathfrak{g}),\\ 0 & \text{if}\quad a\in\mathfrak{D}(\mathfrak{g}).\end{cases}\tag{1.24}$$

Proof

Suppose that G is a Riemannian flat Lie group and $\mathfrak{g}$ its Lie algebra. Let $(\mathfrak{D}(\mathfrak{g})^k)_{k\in\mathbb{N}}$ denote the commutator series of $\mathfrak{g}$ defined recursively by

$$\mathfrak{D}(\mathfrak{g})^0=\mathfrak{g},\quad \mathfrak{D}(\mathfrak{g})^1=\mathfrak{D}(\mathfrak{g})\quad\text{and}\quad \mathfrak{D}(\mathfrak{g})^{k+1}=[\mathfrak{D}(\mathfrak{g})^k,\mathfrak{D}(\mathfrak{g})^k].$$

We deduce from Proposition 1.4.1 that

$$\mathfrak{L}(\mathfrak{g}) = \mathfrak{D}(\mathfrak{g})^{\perp} = N_r(\mathfrak{g}) = (\mathfrak{g}\mathfrak{g})^{\perp}. \tag{1.25}$$

These relations imply that $\mathfrak{L}(\mathfrak{g})$ is abelian, $\mathfrak{D}(\mathfrak{g}) = \mathfrak{g}\mathfrak{g}$ and hence $\mathfrak{D}(\mathfrak{g})$ is a two-sided ideal of the Levi-Civita product so $\mathfrak{D}(\mathfrak{g})$ endowed with the restricted metric is a Riemannian flat Lie algebra and, by induction, for any $k \in \mathbb{N}$, $\mathfrak{D}(\mathfrak{g})^k$ is a Riemannian flat Lie algebra. On the other hand, it is known that a non null left symmetric algebra cannot be equal to its derived ideal (see [9] pp.31). So $\mathfrak{g}$ must be solvable and hence $\mathfrak{D}(\mathfrak{g})$ is nilpotent. If $\mathfrak{D}(\mathfrak{g})$ is non abelian then the splitting

$$\mathfrak{D}(\mathfrak{g}) = \mathfrak{L}(\mathfrak{D}(\mathfrak{g})) \oplus \mathfrak{D}^2(\mathfrak{g})$$

is non trivial. But the center of $\mathfrak{D}(\mathfrak{g})$ is contained in $\mathfrak{L}(\mathfrak{D}(\mathfrak{g}))$ and it intersects non trivially $\mathfrak{D}^2(\mathfrak{g})$ ($\mathfrak{D}(\mathfrak{g})$ is nilpotent) so $\mathfrak{D}(\mathfrak{g})$ must be abelian. This achieves the direct part of the theorem. The equation (1.24) is easy to establish and the converse follows immediately from this equation.
Suppose that G is flat. Hence $\mathfrak{L}(\mathfrak{g})$ is abelian, $\mathfrak{D}(\mathfrak{g})$ is abelian and

$$\mathfrak{g} = \mathfrak{L}(\mathfrak{g}) \oplus \mathfrak{D}(\mathfrak{g}).$$

If $\mathfrak{L}(\mathfrak{g}) = \{0\}$ then $\mathfrak{g} = \mathfrak{D}(\mathfrak{g}) = \{0\}$ and the result follows trivially. Suppose now that $\mathfrak{L}(\mathfrak{g}) \neq \{0\}$. Let $(s_1, ..., s_p)$ be a basis of $\mathfrak{L}(\mathfrak{g})$. The restriction of ad_{s_1} to $\mathfrak{D}(\mathfrak{g})$ is a skew-symmetric endomorphism, thus its kernel K_1 is of even codimension in $\mathfrak{D}(\mathfrak{g})$. Now, ad_{s_2} commutes with ad_{s_1} and K_1 is invariant by ad_{s_2}. By using the same argument as above, we deduce that $K_2 = K_1 \cap \ker \mathrm{ad}_{s_2}$ is of even codimension in K_1. Finally K_2 is of even codimension in $\mathfrak{D}(\mathfrak{g})$. Thus, by induction, we show that

$$K_p = \mathfrak{D}(\mathfrak{g}) \cap (\cap_{i=1}^{p} \ker \mathrm{ad}_{s_i})$$

is an even codimensional subspace of $\mathfrak{D}(\mathfrak{g})$. Now from its definition K_p is contained in the center of $\mathfrak{g}$ which is contained in $\mathfrak{L}(\mathfrak{g})$ and then $K_p = \{0\}$ and the second part of the theorem follows. □

1.4.3 Riemannian Lie groups with negative sectional curvature

The study of Riemannian Lie groups with negative sectional curvature is based on the following result established by Heintze in [7].

Proposition 1.4.2 *A connected homogeneous manifold of non-positive curvature can be represented as a connected solvable Lie group with a left invariant metric.*

From this proposition we deduce, in particular, that a Riemannian Lie group with negative sectional curvature must be solvable. Based on this remark Heintze gave a classification of Lie groups which admits a left invariant metric of strictly negative sectional curvature. The main result of Heintze is the following theorem.

Theorem 1.4.3 *Let G be a solvable Lie group. Then the following conditions are equivalent:*

(i) *G admits a left invariant Riemannian metric of strictly negative curvature.*

(ii) $\dim \mathfrak{D}(\mathfrak{g}) = \dim \mathfrak{g} - 1$, *and there exists $A \in \mathfrak{g}$ such that the eigenvalues of $\mathrm{ad}_{A|\mathfrak{D}(\mathfrak{g})}$ have positive real part.*

We cannot give the proof of this result, it constitutes the object of [7]. However, we can give now a key proposition in [7].

Proposition 1.4.3 *Let G be a solvable Riemannian Lie group with strictly negative sectional curvature. Then the following conditions hold:*

(i) $\dim \mathfrak{D}(\mathfrak{g}) = \dim \mathfrak{g} - 1$.

(ii) *there exists a unit vector u in the orthogonal of $\mathfrak{D}(\mathfrak{g})$ such that $D : \mathfrak{D}(\mathfrak{g}) \longrightarrow \mathfrak{D}(\mathfrak{g})$ is positive definite, where D is the symmetric part of $\mathrm{ad}_{u|\mathfrak{D}(\mathfrak{g})} : \mathfrak{D}(\mathfrak{g}) \longrightarrow \mathfrak{D}(\mathfrak{g})$.*

(iii) *If S is the skew-symmetric part of $\mathrm{ad}_{u|\mathfrak{D}(\mathfrak{g})} : \mathfrak{D}(\mathfrak{g}) \longrightarrow \mathfrak{D}(\mathfrak{g})$, then also $D^2 + DS - SD : \mathfrak{D}(\mathfrak{g}) \longrightarrow \mathfrak{D}(\mathfrak{g})$ is positive definite.*

Proof.

(i) Let $u \in \mathfrak{D}(\mathfrak{g})^{\perp}$ and consider $(\mathrm{ad}_u)_{|u^{\perp}} : u^{\perp} \longrightarrow \mathfrak{D}(\mathfrak{g})$. We will prove that $(\mathrm{ad}_u)_{|u^{\perp}}$ is injective. Let v a vector in the kernel of $(\mathrm{ad}_u)_{|u^{\perp}}$. From proposition 1.3.5, we get

$$\langle \mathrm{K}(u,v)u,v\rangle = \frac{1}{4}|\mathrm{ad}_u^* v|^2.$$

Since $Q<0$ this implies that $v=0$, $(\mathrm{ad}_u)_{|u^{\perp}}$ is injective and hence $\dim \mathfrak{D}(\mathfrak{g}) = \dim \mathfrak{g} - 1$.

(ii) Let u be an unit vector in $\mathfrak{D}(\mathfrak{g})^{\perp}$. Since $\mathfrak{g}$ is solvable, $\mathfrak{D}(\mathfrak{g})$ is nilpotent and hence there exists $z \neq 0$ in the center of $\mathfrak{D}(\mathfrak{g})$. From Proposition 1.3.5, we get for any $v \in \mathfrak{D}(\mathfrak{g})$,

$$\begin{aligned}\langle \mathrm{K}(z,v)z,v\rangle &= \frac{1}{4}|\mathrm{ad}_z^* v + \mathrm{ad}_v^* z|^2 - \langle \mathrm{ad}_z^* z, \mathrm{ad}_v^* v\rangle \\ &= \frac{1}{4}|\mathrm{ad}_z^* v + \mathrm{ad}_v^* z|^2 - \langle z,[z,u]_{\mathfrak{g}}\rangle\langle v,[v,u]_{\mathfrak{g}}\rangle,\end{aligned}$$

since from $\mathfrak{g} = \mathfrak{D}(\mathfrak{g}) \oplus \mathbb{R}u$, we get

$$\mathrm{ad}_z^* z = \langle z,[z,u]_{\mathfrak{g}}\rangle u \quad \text{and} \quad \mathrm{ad}_v^* v = \langle v,[v,u]_{\mathfrak{g}}\rangle u + q(v),$$

where $q(v) \in \mathfrak{D}(\mathfrak{g})$. Let D be the symmetric part of $\mathrm{ad}_{u|\mathfrak{D}(\mathfrak{g})}$. Then

$$\langle \mathrm{K}(z,v)z,v\rangle = \frac{1}{4}|\mathrm{ad}_z^* v + \mathrm{ad}_v^* z|^2 - \langle Dz,z\rangle\langle Dv,v\rangle.$$

Suppose that $v \in \ker D$. From the relation above and the fact that $Q<0$ we must have $v=\lambda z$. If $\lambda \neq 0$ then $[u,z]_{\mathfrak{g}} = 0$ which implies that z is in the center of $\mathfrak{g}$. But we have seen in Proposition 1.3.5 that if $Q<0$ the center of $\mathfrak{g}$ is trivial so $z=0$ which is impossible. Thus D is an isomorphism. From the expression of $\langle \mathrm{K}(z,v)z,v\rangle$ given above and the fact that $Q<0$ we deduce that D is either definite positive or definite negative, so by replacing u by $-u$ we get the assertion.

(iii) Fix $u \in \mathfrak{g}$ as in (ii). We have, for any $v \in \mathfrak{D}(\mathfrak{g})$, $[u,v] = Dv + Sv$. From

Proposition 1.3.5 and since $\mathrm{ad}_u^*u = \mathrm{ad}_v^*u = 0$, we get for any $v \in \mathfrak{D}(\mathfrak{g}) \setminus \{0\}$,

$$\begin{aligned}\langle \mathrm{K}(u,v)u, v\rangle &= -\frac{3}{4}|\mathrm{ad}_u v|^2 + \frac{1}{4}|\mathrm{ad}_u^* v|^2 - \frac{1}{2}\langle \mathrm{ad}_u v, \mathrm{ad}_u^* v\rangle \\ &= -\frac{3}{4}|Dv + Sv|^2 + \frac{1}{4}|Dv - Sv|^2 - \frac{1}{2}\langle Dv + Sv, Dv - Sv\rangle \\ &= -\langle Dv, Dv\rangle - 2\langle Dv, Sv\rangle \\ &= -\langle (D^2 - (SD - DS))v, v\rangle.\end{aligned}$$

This achieves the proof. □

The Riemannian Lie groups with $Q \leq 0$ have been classified by Azencott and Wilson [3]. Since the statements are complicated we will content ourselves with the following qualitative result.

Theorem 1.4.4 *If a connected Lie group G has a left invariant metric with all sectional curvatures $Q \leq 0$, the it is solvable. If G is unimodular, then any such metric with $Q \leq 0$ must actually be flat ($Q \equiv 0$).*

Remark 2 *To our knowledge, unlike the negative case, there is no systematic study of Riemannian Lie group with positive sectional curvature.*

1.5 Ricci curvature of Riemannian Lie groups

We start this section by this general result.

Theorem 1.5.1 *Suppose that the Lie algebra of G is nilpotent but not abelian. Then for any left invariant metric on G the scalar curvature is strictly negative and there exists a direction of strictly negative Ricci curvature and a direction of strictly positive Ricci curvature.*

Proof. From (1.21) and Proposition 1.3.7, we have $\mathfrak{s} = -\frac{1}{4}\mathrm{tr}\mathcal{J}_1 < 0$.
Since $\mathfrak{s} < 0$ there exists $v \in \mathfrak{g}$ such that $\mathrm{ric}(v,v) < 0$. On the other hand, since $\mathfrak{g}$ is nilpotent there exists a non null vector $u \in \mathfrak{D}(\mathfrak{g}) \cap Z(\mathfrak{g})$. From Proposition 1.3.6, we get $\mathrm{ric}(u,u) = -\frac{1}{4}\mathrm{tr}(\mathrm{J}_u^2)$. Since J_u is skew-adjoint $\mathrm{tr}(\mathrm{J}_u^2) \leq 0$ and $\mathrm{tr}(\mathrm{J}_u^2) = 0$ if and only if $\mathrm{J}_u = 0$. This is equivalent by Proposition 1.3.7 to $u \in \mathfrak{D}(\mathfrak{g})^\perp$ which is impossible. □

1.5.1 Riemannian Lie groups with positive Ricci curvature metrics

Proposition 1.5.1 *If a connected Lie group admits a left invariant metric with all Ricci curvatures positive then it must be unimodular.*

Proof. Since $H \in \mathfrak{D}(\mathfrak{g})^\perp$, $\mathrm{J}_H = 0$ and then from Proposition 1.3.6 we get

$$\mathrm{ric}(H, H) = -\frac{1}{4}\mathrm{tr}((\mathrm{ad}_H + \mathrm{ad}_H^*)^2). \tag{1.26}$$

Since $\mathrm{tr}(\mathrm{ad}_H) = |H|^2$ we get $\mathrm{ad}_H + \mathrm{ad}_H^* = 0$ iff $H = 0$. Thus if $H \neq 0$ then $\mathrm{ric}(H, H) < 0$. □

Theorem 1.5.2 *A connected Lie group admits a left invariant metric with all Ricci curvatures strictly positive if and only if it is compact with finite fundamental group.*

Proof. The direct sense follows from Myers's Theorem (see Theorem 1.2.3). In the other direction, if G is compact then we can choose a bi-invariant metric on G which Ricci curvature, according to Proposition 1.3.10, is given by

$$\mathrm{ric}(u, u) = \frac{1}{4}\mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_u^*) \geq 0,$$

and $\mathrm{ric}(u, u) = 0$ if and only if $u \in Z(\mathfrak{g})$. Now the hypothesis that G has a finite fundamental group implies that its universal covering is also compact and hence $Z(\mathfrak{g}) = \{0\}$. So all Ricci curvatures are strictly positive. □

A proof of the following theorem can be found in [2]. It is based on a geometric theorem. It can be interesting to find an algebraic proof of this result.

Theorem 1.5.3 *A left invariant Riemannian metric on a Lie group is Ricci flat if and only if it is flat.*

1.5.2 Riemannian Lie groups with negative Ricci curvature metrics

It remains open the classification of Lie groups admitting left invariant metrics with all Ricci curvatures < 0 or ≤ 0. In [6] we have the following result.

Theorem 1.5.4 *A solvable unimodular Lie group G with a left invariant metric has all Ricci curvatures ≤ 0 if and only if its derived Lie algebra $\mathfrak{D}(\mathfrak{g})$ is abelian,*

$\mathfrak{D}(\mathfrak{g})^{\perp}$ *is a commutative subalgebra and the linear transformation* ad_b *is normal for every* $b \in \mathfrak{D}(\mathfrak{g})^{\perp}$.

To prove this theorem, we need the following lemma.

Lemma 1.5.1 *Let* $(\mathfrak{g}, \langle\ ,\ \rangle)$ *be an unimodular Lie algebra such that* $\mathfrak{g}$ *splits orthogonally as*

$$\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{b},$$

where $\mathfrak{u}$ *is an abelian ideal and* $\mathfrak{b}$ *is an abelian subalgebra. Then*

(i) *Given an orthonormal basis* $b_1, \ldots, b_r$ *of* $\mathfrak{b}$ *and* $u \in \mathfrak{u}$, *we have*

$$\mathrm{Ric}(u) = \frac{1}{2}[L_i, L_i^*](u),$$

where L_i *is the restriction of* ad_{b_i} *to* $\mathfrak{u}$.

(ii) *The self-adjoint transformation* $\mathrm{Ric} \equiv 0$ *on* $\mathfrak{u}$ *if and only if, for any* $b \in \mathfrak{b}$, *the restriction of* ad_b *to* $\mathfrak{u}$ *is a normal transformation.*

Proof.

(i) Since G is solvable, the Cartan's criteria of solvability (see [10]) implies that for any $u \in \mathfrak{D}(\mathfrak{g})$ and any $v \in \mathfrak{g}$, $B(u,v) = 0$. So, by (1.20), for any $u \in \mathfrak{D}(\mathfrak{g})$ and any $v \in \mathfrak{g}$,

$$\mathrm{ric}(u,v) = -\frac{1}{2}\langle \mathcal{J}_1 u, v\rangle + \frac{1}{4}\langle \mathcal{J}_2 u, v\rangle.$$

Choose an orthonormal basis $(e_1, \ldots, e_p)$ an orthonormal basis of $\mathfrak{D}(\mathfrak{g})$. From Proposition 1.3.7, we have

$$\begin{aligned} \mathcal{J}_1 u &= \sum_{i=1}^{p} \mathrm{ad}_{e_i}^* \circ \mathrm{ad}_{e_i} u + \sum_{i=1}^{p} \mathrm{ad}_{b_i}^* \circ \mathrm{ad}_{b_i} u \\ &= \sum_{i=1}^{p} \mathrm{ad}_{b_i}^* \circ \mathrm{ad}_{b_i} u, \\ \mathcal{J}_2 u &= \sum_{i=1}^{p} \mathrm{ad}_{e_i} \circ \mathrm{ad}_{e_i}^* u + \sum_{i=1}^{p} \mathrm{ad}_{b_i} \circ \mathrm{ad}_{b_i}^* u. \end{aligned}$$

Now

$$\begin{aligned} \mathrm{ad}^*_{e_i}u &= \sum_{j=1}^{r}\langle u,[e_i,b_j]_{\mathfrak{g}}\rangle b_j, \\ \mathrm{ad}^*_{b_i}u &= \sum_{j=1}^{p}\langle u,[b_i,e_j]_{\mathfrak{g}}\rangle e_j. \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{Ric}(u) &= -\frac{1}{2}\sum_{i=1}^{p}\mathrm{ad}^*_{b_i}\circ\mathrm{ad}_{b_i}u+\frac{1}{4}\sum_{i,j}\langle u,[e_i,b_j]_{\mathfrak{g}}\rangle[e_i,b_j]_{\mathfrak{g}}+\frac{1}{4}\sum_{i,j}\langle u,[b_i,e_j]_{\mathfrak{g}}\rangle[b_i,e_j]_{\mathfrak{g}} \\ &= -\frac{1}{2}\sum_{i=1}^{p}\mathrm{ad}^*_{b_i}\circ\mathrm{ad}_{b_i}u+\frac{1}{2}\sum_{i,j}\langle u,[e_i,b_j]_{\mathfrak{g}}\rangle[e_i,b_j]_{\mathfrak{g}} \\ &= -\frac{1}{2}\sum_{i=1}^{p}\mathrm{ad}^*_{b_i}\circ\mathrm{ad}_{b_i}u+\frac{1}{2}\sum_{j}[b_j,\sum_{i}\langle u,[b_j,e_i]_{\mathfrak{g}}\rangle e_i]_{\mathfrak{g}} \\ &= -\frac{1}{2}\sum_{i=1}^{p}\mathrm{ad}^*_{b_i}\circ\mathrm{ad}_{b_i}u+\frac{1}{2}\sum_{j}[b_j,\sum_{i}\langle \mathrm{ad}^*_{b_j}u,e_i\rangle e_i]_{\mathfrak{g}} \\ &= \frac{1}{2}\sum_{j}[\mathrm{ad}_{b_i},\mathrm{ad}^*_{b_i}](u). \end{aligned}$$

(ii) This is a consequence of the following property. "Let $L_1,\dots,L_r$ be a family of commuting endomorphisms on $\mathbb{R}^n$ such that $\sum_i[L_i,L_i^*]=0$, where L_i^* is the adjoint of L_i with respect to an inner definite positive product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$. Then $L_1,\dots,L_r$ are normal transformation". We will prove this property by induction on n. The property is obviously true for $n=1$. Since the L_i commute, the associated transformation $\widetilde{L}_i$ on $\mathbb{C}^n$ have a common eigenvector x with eigenvalues $\lambda_1,\dots,\lambda_r$. The new family $\widetilde{L}_i-\lambda_i I$ satisfies

$$\sum_{i=1}^{r}[\widetilde{L}_i-\lambda_i\mathrm{I},(\widetilde{L}_i-\lambda_i\mathrm{I})^*]=0.$$

(Here we are considering $\mathbb{C}^n$ with the Hermitian product associated to $\langle\ ,\ \rangle$).

Then

$$\begin{aligned} 0 &= \sum_{i=1}^{r} \langle [\widetilde{L}_i - \lambda_i \mathrm{I}, (\widetilde{L}_i - \lambda_i \mathrm{I})^*] x, x \rangle \\ &= \sum_{i=1}^{r} |(\widetilde{L}_i - \lambda_i \mathrm{I})^*|^2. \end{aligned}$$

This shows that x is a common eigenvector of $\widetilde{L}_i^*$. Therefore, we may restrict each $\widetilde{L}_i$ to the the orthogonal complement of x and use the induction hypothesis to conclude. □

Proof of Theorem 1.5.4. Since G is solvable, the Cartan's criteria of solvability (see [10]) implies that for any $u \in \mathfrak{D}(\mathfrak{g})$ and any $v \in \mathfrak{g}$, $B(u,v) = 0$. Moreover, $\mathfrak{D}(\mathfrak{g})$ is nilpotent and hence its center, say Z_0, is non trivial with $Z_0 \cap [\mathfrak{D}(\mathfrak{g}), \mathfrak{D}(\mathfrak{g})] \neq \{0\}$. Put

$$\mathfrak{g} = Z_0 \oplus \mathfrak{h} \oplus \mathfrak{D}(\mathfrak{g})^{\perp},$$

and choose an orthonormal basis $(z_1, \ldots, z_p, e_1, \ldots, e_q, f_1, \ldots, f_r)$ respecting this splitting. From (1.20) we get, for any $z \in Z_0$,

$$\mathrm{ric}(z,z) = -\frac{1}{2}\langle \mathcal{J}_1 z, z\rangle + \frac{1}{4}\langle \mathcal{J}_2 z, z\rangle.$$

By using the Jacobi identity, one can see easily that ad_{f_i} preserves Z, we get for any $i = 1, \ldots, r$,

$$[f_i, z]_{\mathfrak{g}} = \sum_{j=1}^{p} \langle [f_i, z]_{\mathfrak{g}}, z_j \rangle z_j.$$

From Proposition 1.3.7, we have

$$\begin{aligned} \langle \mathcal{J}_1 z, z\rangle &= \sum_{i=1}^{r} \langle [f_i, z]_{\mathfrak{g}}, [f_i, z]_{\mathfrak{g}} \rangle, \\ &= \sum_{i,j} \langle [f_i, z]_{\mathfrak{g}}, z_j \rangle^2, \\ \langle \mathcal{J}_2 z, z\rangle &= \sum_{i=1}^{p} \langle \mathrm{ad}_{z_i}^* z, \mathrm{ad}_{z_i}^* z\rangle + \sum_{i=1}^{p} \langle \mathrm{ad}_{e_i}^* z, \mathrm{ad}_{e_i}^* z\rangle + \sum_{i=1}^{p} \langle \mathrm{ad}_{f_i}^* z, \mathrm{ad}_{f_i}^* z\rangle. \end{aligned}$$

Now

$$\begin{aligned}
\mathrm{ad}^*_{z_i} z &= \sum_{j=1}^{r} \langle [z_i, f_j], z\rangle f_j,\\
\mathrm{ad}^*_{e_i} z &= \sum_{j=1}^{q} \langle [e_i, e_j], z\rangle e_j + \sum_{j=1}^{r} \langle [e_i, f_j], z\rangle f_j,\\
\mathrm{ad}^*_{f_i} z &= \sum_{j=1}^{p} \langle [f_i, z_j], z\rangle z_j + \sum_{j=1}^{q} \langle [f_i, e_j], z\rangle e_j + \sum_{j=1}^{r} \langle [f_i, f_j], z\rangle f_j.
\end{aligned}$$

So we get

$$r(z,z) = -\frac{1}{2}\sum_{i,j}\langle [f_i, z]_{\mathfrak{g}}, z_j\rangle^2 + \frac{1}{2}\sum_{i,j}\langle [z_i, f_j], z\rangle^2 + \frac{1}{2}\sum_{i,j}\langle [f_i, e_j], z\rangle^2 + \frac{1}{4}\sum_{i,j}\langle [e_i, e_j], z\rangle^2 + \frac{1}{4}\sum_{i,j}\langle [f_i, f_j], z\rangle^2.$$

Now since ric ≤ 0, we get

$$0 \geq \sum_k r(z_k, z_k) = \frac{1}{2}\sum_{i,j,k}\langle [f_i, e_j], z_k\rangle^2 + \frac{1}{4}\sum_{i,j,k}\langle [e_i, e_j], z_k\rangle^2 + \frac{1}{4}\sum_{i,j,k}\langle [f_i, f_j], z_k\rangle^2.$$

This shows that, for any i, j, k,

$$\langle [f_i, e_j], z_k\rangle = \langle [e_i, e_j], z_k\rangle = \langle [f_i, f_j], z_k\rangle = 0.$$

This shows that Z_0 is orthogonal to $[\mathfrak{D}(\mathfrak{g}), \mathfrak{D}(\mathfrak{g})] = 0$, so $Z_0 = \mathfrak{D}(\mathfrak{g})$. We deduce also that $\mathfrak{D}(\mathfrak{g})^\perp$ is also abelian. We have shown also that for any $u, v \in \mathfrak{D}(\mathfrak{g})$, $\mathrm{ric}(u, v) = \langle \mathrm{Ric} u, v\rangle = 0$. From Lemma 1.5.1 (i) we have also that Ric preserves $\mathfrak{D}(\mathfrak{g})$ so for any $u \in \mathfrak{D}(\mathfrak{g})$, $\mathrm{Ric} u = 0$. Hence from Lemma (ii), we can achieve the direct sense of the theorem.

Conversely, if $\mathfrak{g} = \mathfrak{D}(\mathfrak{g}) \oplus \mathfrak{D}(\mathfrak{g})^\perp$, where $\mathfrak{D}(\mathfrak{g})$ and $\mathfrak{D}(\mathfrak{g})^\perp$ are abelian and ad_b is normal for any $b \in \mathfrak{D}(\mathfrak{g})^\perp$, we get from Lemma 1.5.1 that for any $u \in \mathfrak{D}(\mathfrak{g})$ and $b \in \mathfrak{D}(\mathfrak{g})^\perp$,

$$\mathrm{ric}(u + b, v + b) = \mathrm{ric}(b, b) = -\frac{1}{2}\mathrm{tr}((\mathrm{ad}_b + \mathrm{ad}^*_b)^2) \leq 0.$$

This achieve the proof of the theorem. □

We can deduce from Theorem 1.5.4 and its proof the following corollary.

Corollary 1.5.1 *No unimodular solvable Lie group carries a left invariant Riemannian metric with* ric < 0.

Moreover, we can combine the proof of Theorem 1.5.4, Proposition 1.5.1 and Theorem 1.4.2 to get the following result which is a particular case of Theorem 1.5.3.

Proposition 1.5.2 *A left invariant Riemannian metric on a solvable Lie group is Ricci flat if and only if it is flat.*

1.6 Scalar curvature of Riemannian Lie groups

Theorem 1.6.1 *Let* $(\mathfrak{g}, \langle\ ,\ \rangle)$ *be a solvable Lie algebra. Then*

$$2\mathrm{t}r(\widehat{B}) + \mathrm{t}r(\mathcal{J}_1) \geq 0$$

and the equality holds if and only if $\langle\ ,\ \rangle$ *is flat.*

Proof. Note first that by Cartan's criteria for solvability (See [10] pp. 50) we have $\widehat{B}u = 0$ for any $u \in \mathfrak{D}(\mathfrak{g})$. We consider an orthonormal basis $\mathbb{B} = \{e_1, \ldots, e_p, f_1, \ldots, f_q\}$ of $\mathfrak{g}$ where $\{e_1, \ldots, e_p\}$ is a basis of $\mathfrak{D}(\mathfrak{g})$ and $\{f_1, \ldots, f_q\}$ is a basis of $\mathfrak{D}(\mathfrak{g})^\perp$. We have, according to Proposition 1.3.7,

$$\begin{aligned}
\mathrm{tr}(\mathcal{J}_1) &= \sum_{i=1}^{p} \mathrm{tr}\left(\mathrm{ad}_{e_i} \circ \mathrm{ad}_{e_i}^*\right) + \sum_{i=1}^{q} \mathrm{tr}\left(\mathrm{ad}_{f_i} \circ \mathrm{ad}_{f_i}^*\right), \\
2\mathrm{tr}(\widehat{B}) &= 2\sum_{i=1}^{q} \mathrm{tr}\left(\mathrm{ad}_{f_i} \circ \mathrm{ad}_{f_i}\right).
\end{aligned}$$

For $i = 1, \ldots, p$ and $j = 1, \ldots, q$, we denote by M_i and N_j the matrix of ad_{e_i} and ad_{f_j} in $\mathbb{B}$ respectively. We have

$$M_i = \begin{pmatrix} A_i & X_i \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad N_j = \begin{pmatrix} B_j & Y_j \\ 0 & 0 \end{pmatrix},$$

where M_i and N_j are p-square matrix and X_i and Y_j are (p,q)-matrix. We have

$$\begin{aligned} \mathrm{tr}(\mathcal{J}_1) &= \sum_{i=1}^{p}\left(\mathrm{tr}\left(A_iA_i^t\right)+\mathrm{tr}\left(X_iX_i^t\right)\right)+\sum_{j=1}^{q}\left(\mathrm{tr}\left(B_jB_j^t\right)+\mathrm{tr}\left(Y_jY_j^t\right)\right), \\ 2\mathrm{tr}(\widehat{B}) &= 2\sum_{i=1}^{q}\mathrm{tr}\left(B_i^2\right). \end{aligned}$$

The key formula is

$$\sum_{i=1}^{p} tr\left(X_iX_i^t\right)=\sum_{j=1}^{q}\mathrm{tr}(B_jB_j^t)$$

which implies that

$$2\mathrm{tr}(\widehat{B})+\mathrm{tr}(\mathcal{J}_1) = \sum_{i=1}^{p}\mathrm{tr}\left(A_iA_i^t\right)+\sum_{j=1}^{q}\mathrm{tr}\left(Y_jY_j^t\right)+\sum_{j=1}^{q}\mathrm{tr}(B_j+B_j^t)^2.$$

This formula shows that $2tr(\widehat{B})+tr(\mathcal{J}_1)\geq 0$ and equality occurs when $A_i=0$, $Y_j=0$ and $B_j+B_j^t=0$ for $i=1,\ldots,p$ and $j=1,\ldots,q$. This means that $\mathfrak{D}(\mathfrak{g})$ and $\mathfrak{D}(\mathfrak{g})^\perp$ are both abelian and ad_u is skew-adjoint for any $u\in\mathfrak{D}(\mathfrak{g})^\perp$. According to Theorem 1.4.2, this is equivalent to that $\langle\ ,\ \rangle$ is flat. □

An important consequence of this theorem is a new proof of the following result du to Heintze and Jensen.

Theorem 1.6.2 *Let G be a Riemannian non flat solvable Lie group. Then*

$$\mathfrak{s}=-|H|^2-\frac{1}{4}\left(2tr(\widehat{B})+tr(\mathcal{J}_1)\right)<0.$$

Theorem 1.6.3 *If the Lie algebra of G is non abelian then G possesses a left invariant metric of strictly negative scalar curvature.*

Proof. First suppose that there exists linearly independent vectors x,y,z in $\mathfrak{g}$ such that $[x,y]=z$ and choose a basis $b_1,\ldots,b_n$ such that $b_1=x$, $b_2=y$ and $b_3=z$. For any real number $\epsilon>0$, consider an auxiliary basis $e_1,\ldots,e_n$ defined by $e_i=a_ib_i$ with $(a_1,\ldots,a_n)=(\epsilon,\epsilon,\epsilon^2,\ldots,\epsilon^2)$. Define a left invariant metric by requiring that $(e_1,\ldots,e_n)$ should be orthonormal. Let $\mathfrak{g}^\epsilon$ denote the Lie algebra provided with this particular metric and this particular orthonormal basis. Sitting $[e_i, e_j]_{\mathfrak{g}} = \sum \alpha_{ij}^k e_k$, the structure constants are clearly function of ϵ. Indeed if $[b_i, b_j]_{\mathfrak{g}} = \sum C_{ij}^k b_k$,

$$[e_i, e_j]_{\mathfrak{g}} = a_i a_j [b_i, b_j] = \sum C_{ij}^k a_i a_j b_k = \sum \frac{a_i a_j}{a_k} C_{ij}^k e_k,$$

thus, for any $1 \leq i, j, k \leq n$,

$$\alpha_{ij}^k = \frac{a_i a_j}{a_k}.$$

From this relation and the definition of the a_i, one can see that each α_{ij}^k tends to a well define limit. Thus we obtain a limit algebra $\mathfrak{g}^0$ with a prescribed metric and prescribed orthonormal basis. One can see easily that the bracket on $\mathfrak{g}^0$ is given by

$$[e_1, e_2]_{\mathfrak{g}}^0 = e_3$$

with $[e_i, e_j]_{\mathfrak{g}}^0 = 0$ otherwise. Thus $\mathfrak{g}^0$ is nilpotent and hence, from Proposition 1.3.8, its scalar curvature $\mathfrak{s}(\mathfrak{g}^0) < 0$. It follows by continuity that $\mathfrak{s}(\mathfrak{g}^\epsilon) < 0$ whenever ϵ is sufficiently close to 0.

On the other hand, if x, y and $[x, y]_{\mathfrak{g}}$ are always linearly independent, then $\mathfrak{g}$ is isomorphic to the special example in 2, hence $\mathfrak{g}$ has strictly negative curvature for any choice of metric. □

Theorem 1.6.4 *If the universal covering of G is not homeomorphic to Euclidean* (*or if G contains a compact non-commutative subgroup*) *then G admits a left invariant metric of strictly positive scalar curvature.*

Proof. See [13]. □

1.7 The 3-dimensional case

In this section, we study the curvatures of left invariant Riemannian metric on 3-dimensional Lie groups. The main tool to be used is the Euclidean cross product. If V is a three dimensional vector space endowed with a positive definite metric and a fixed orientation, then there exists a bilinear map $V \times V \longrightarrow V$ called cross product satisfying:

1. for any $u, v \in V$, $u \times v = -v \times u$ is orthogonal to u and v and $|u \times v| = \sqrt{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}$,

2. if u, v are linearly independent then $(u, v, u \times v)$ is a positively oriented basis of V.

Let G be a connected 3-dimensional Lie group with a left invariant Riemannian metric. Choose an orientation of $\mathfrak{g}$ such that the cross product is definite. The following lemma plays a crucial role in the the study of the curvature of G.

Lemma 1.7.1 *There exists a unique endomorphism $L : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that, for any $u, v \in \mathfrak{g}$,*

$$[u, v]_{\mathfrak{g}} = L(u \times v).$$

Moreover, G is unimodular if and only if $L^* = L$.

Proof. Choose an oriented orthonormal basis $(e_1, e_2, , e_3)$ of $\mathfrak{g}$ and put

$$L(e_1) = [e_2, e_3]_{\mathfrak{g}},\ L(e_2) = [e_3, e_1]_{\mathfrak{g}} \quad \text{and} \quad L(e_3) = [e_1, e_2]_{\mathfrak{g}}.$$

It is easy now to check that L satisfies the desired property. Moreover, if $L(e_i) = a_{1i}e_1 + a_{2i}e_2 + a_{3i}e_3$ then

$$\mathrm{tr}(\mathrm{ad}_{e_1}) = a_{23} - a_{32},\ \mathrm{tr}(\mathrm{ad}_{e_2}) = -a_{13} + a_{31} \quad \text{and} \quad \mathrm{tr}(\mathrm{ad}_{e_3}) = a_{12} - a_{21}.$$

So G is unimodular if and only if the matrix (a_{ij}) is symmetric or $L^* = L$. □

Let us specialize to the unimodular case. If L is self-adjoint there exists an oriented orthonormal basis (e_1, e_2, e_3) such that $L(e_i) = \lambda_i e_i$ with $\lambda_1 \leq \lambda_2 \leq \lambda_3$. Thus

$$[e_1, e_2]_{\mathfrak{g}} = \lambda_3 e_3,\ [e_2, e_3]_{\mathfrak{g}} = \lambda_1 e_1 \quad \text{and} \quad [e_3, e_1]_{\mathfrak{g}} = \lambda_2 e_2. \tag{1.27}$$

The three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are well defined up to order. However, the construction is based on a choice of orientation. If we reverse the orientation, L will changes sign and $\lambda_1, \lambda_2, \lambda_3$ also.

Let compute the Ricci and scalar curvature. We will use the formula

$$\mathrm{ric}(u,v) = -\frac{1}{2}B(u,v) - \frac{1}{2}\mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v^*) - \frac{1}{4}\mathrm{tr}(\mathrm{J}_u \circ \mathrm{J}_v) - \frac{1}{2}(\langle \mathrm{ad}_H u, v\rangle + \langle \mathrm{ad}_H v, u\rangle).$$

established in Proposition 1.3.6. Since G is unimodular, $H = 0$. Now, by identifying an endomorphism with its matrix in the basis (e_1, e_2, e_3), we get

$$\mathrm{ad}_{e_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\lambda_2 \\ 0 & \lambda_3 & 0 \end{pmatrix}, \ \mathrm{ad}_{e_2} = \begin{pmatrix} 0 & 0 & \lambda_1 \\ 0 & 0 & 0 \\ -\lambda_3 & 0 & 0 \end{pmatrix}, \ \mathrm{ad}_{e_3} = \begin{pmatrix} 0 & -\lambda_1 & 0 \\ \lambda_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{ad}_{e_1}^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda_3 \\ 0 & -\lambda_2 & 0 \end{pmatrix}, \ \mathrm{ad}_{e_2}^* = \begin{pmatrix} 0 & 0 & -\lambda_3 \\ 0 & 0 & 0 \\ \lambda_1 & 0 & 0 \end{pmatrix}, \ \mathrm{ad}_{e_3}^* = \begin{pmatrix} 0 & \lambda_2 & 0 \\ -\lambda_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From the definition of J_u, we have

$$\mathrm{J}_u v = \langle u, [v, e_1]_{\mathfrak{g}}\rangle e_1 + \langle u, [v, e_2]_{\mathfrak{g}}\rangle e_2 + \langle u, [v, e_3]_{\mathfrak{g}}\rangle e_3,$$

and by a direct computation, we get

$$\mathrm{J}_{e_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\lambda_1 \\ 0 & \lambda_1 & 0 \end{pmatrix}, \ \mathrm{J}_{e_2} = \begin{pmatrix} 0 & 0 & \lambda_2 \\ 0 & 0 & 0 \\ -\lambda_2 & 0 & 0 \end{pmatrix}, \ \mathrm{J}_{e_3} = \begin{pmatrix} 0 & -\lambda_3 & 0 \\ \lambda_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By using the expression of the Ricci curvature given above, we get

$$(\mathrm{ric}(e_i, e_j)) = \begin{pmatrix} \lambda_2\lambda_3 + \frac{1}{2}(\lambda_1^2 - \lambda_2^2 - \lambda_3^2) & 0 & 0 \\ 0 & \lambda_3\lambda_1 + \frac{1}{2}(\lambda_2^2 - \lambda_1^2 - \lambda_3^2) & 0 \\ 0 & 0 & \lambda_1\lambda_2 + \frac{1}{2}(\lambda_3^2 - \lambda_1^2 - \lambda_3^2) \end{pmatrix}.$$

Put $\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i$. Then

$$(\mathrm{ric}(e_i, e_j)) = 2 \begin{pmatrix} \mu_2\mu_3 & 0 & 0 \\ 0 & \mu_3\mu_1 & 0 \\ 0 & 0 & \mu_1\mu_2 \end{pmatrix}, \tag{1.28}$$

and the scalar curvature is given by

$$\mathfrak{s} = 2(\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3).$$

To compute the sectional curvature, we can use the following formula which holds in any 3-dimensional manifold (see [5] pp. 48):

$$Q(u, v) = \frac{1}{2}|u \times v|^2\mathfrak{s} - \mathrm{ric}(u \times v, u \times v).$$

From what above we can deduce easily the following result.

Proposition 1.7.1 *In the 3-dimensional unimodular case, the determinant* $8(\mu_1\mu_2\mu_3)^2$ *of* ric *is always* ≥ 0*. If this determinant is zero, then at least two of the principal Ricci curvatures must be zero.*

Let us give now a precise description of the possible group according to the signs of $\lambda_1, \lambda_2, \lambda_3$. By changing signs if necessary, we can assume that at most one of the constant structure $\lambda_1, \lambda_2, \lambda_3$ is ≤ 0. So get six distinct cases we tabulate as follows.

Signs of $\lambda_1, \lambda_2, \lambda_3$	Associated Lie group	Description
+,+,+	SU(2) or SO(3)	compact, simple
+,+,−	SL(2, $\mathbb{R}$) or O(1, 2)	noncompact, simple
+,+,0	E(2)	solvable
+,−,0	E(1, 1)	solvable
+,0,0	Heisenberg group	nilpotent
0,0,0	$\mathbb{R}^3$	commutative

It is not difficult that these six possibilities do really give rise to nonisomorphic Lie algebras. They can be distinguished for example by computing the signature of

Killing form. We have

1. SU(2) is the group of 2×2 unitary matrices of determinant 1; homeomorphic to the unit 3-sphere.

2. SO(3) is the rotation group of $\mathbb{R}^3$, isomorphic to $\mathrm{SU}(2)/\{\mathrm{I}_3, -\mathrm{I}_3\}$.

3. $\mathrm{SL}(2,\mathbb{R})$ is the group of 2×2 real matrices of determinant 1.

4. $\mathrm{O}(1,2)$ is Lorentz group consisting of linear transformations the quadratic from $t^2 - x^2 - y^2$. Its identity component is isomorphic to $\mathrm{SL}(2,\mathbb{R})/\{\mathrm{I}_3, -\mathrm{I}_3\}$.

5. E(2) is the group of rigid motion of the Euclidean 2-space.

6. $\mathrm{E}(1,1)$ is the group of rigid motion of the Minkowski 2-space. This group is a semi-direct product of subgroup isomorphic to $\mathbb{R}^2$ and to $\mathbb{R}$, where each $\tau \in \mathbb{R}$ acts on $\mathbb{R}^2$ by the matrix $\begin{pmatrix} e^{\tau} & 0 \\ 0 & e^{-\tau} \end{pmatrix}$.

7. Finally, the Heisenberg group can be described as the group of all 3×3 real matrices of the form $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$.

Let study now how the curvature can be altered by a change of metric.

Proposition 1.7.2 *Depending on the choice of left invariant metric, the Ricci curvature for* SU(2) *can have signature either* $(+,+,+)$ *or* $(+,0,0)$ *or* $(+,-,-)$ *and the scalar curvature can be either positive, negative or zero.*

Proof. This follows from (1.28). □

As in the commutative case all the left invariant metrics all flat, in the nilpotent case, the curvature properties are independent from the metric.

Proposition 1.7.3 *For any metric on the Heisenberg group, the Ricci form has signature* $(+,-,-)$ *and the scalar curvature* $\mathfrak{s}$ *is strictly negative. Furthermore, we have*

$$|\mathrm{ric}(e_1, e_1)| = |\mathrm{ric}(e_2, e_2)| = |\mathrm{ric}(e_3, e_3)| = |\mathfrak{s}|.$$

Proof. We have $\lambda_1 > 0$, $\lambda_2 = \lambda_3 = 0$. So $\mu_1 = -\mu_2 = -\mu_3 = -\frac{1}{2}\lambda_1$. So

$$\mathrm{ric}(e_1, e_1) = -\mathrm{ric}(e_2, e_2) = -\mathrm{ric}(e_3, e_3) = -\mathfrak{s} = \frac{1}{2}\lambda_1^2.$$

The simple group $\mathrm{SL}(2, \mathbb{R})$ and the solvable group $\mathrm{E}(1, 1)$ are difficult to distinguish by curvature properties.

Proposition 1.7.4 *Let G be either* $\mathrm{SL}(2, \mathbb{R})$ *or* $\mathrm{E}(1, 1)$. *Then depending on the choice of left invariant metric the signature of the Ricci form can be either* $(+, -, -)$ *or* $(0, 0, -)$. *However, the scalar curvature $\mathfrak{s}$ must always be strictly negative.*

Proof. If $\lambda_1 = 0$ while λ_2, λ_3 have opposite sign, then $\mathfrak{s} = -\frac{1}{2}(\lambda_2 - \lambda_3)^2 < 0$. If the λ_i are all non zero say $\lambda_1 < 0 < \lambda_2, \lambda_3$, then

$$\frac{\partial \mathfrak{s}}{\partial \lambda_1} = -\lambda_1 + \lambda_2 + \lambda_3 \leq 0$$

shows that $\mathfrak{s}$ is strictly monotone as a function of λ_1 (keeping λ_2, λ_3 fixed) for $\lambda_1 \leq 0$. Therefore

$$\mathfrak{s}(\lambda_1, \lambda_2, \lambda_3) < \mathfrak{s}(0, \lambda_2, \lambda_3) = -\frac{1}{2}(\lambda_2 - \lambda_3)^2 < 0.$$

Further details will be left to the reader. □

Proposition 1.7.5 *The Euclidean group* $\mathrm{E}(2)$ *is non-commutative, but admits a flat left invariant metric. Every nonflat left invariant metric has Ricci form of signature* $(+, -, -)$ *with scalar curvature* $\mathfrak{s} < 0$.

Proof. Left to the reader. □

Now let return to the nonunimodular case. The possible algebras can be described as follows.

Lemma 1.7.2 *If the connected 3-dimensional G is nonunimodular then its Lie algebra has a basis* (e_1, e_2, e_3) *so that*

$$[e_1, e_2]_{\mathfrak{g}} = \alpha e_2 + \beta e_3,\ [e_1, e_3]_{\mathfrak{g}} = \gamma e_2 + \delta e_3 \quad \textit{and} \quad [e_2, e_3]_{\mathfrak{g}} = 0,$$

and so that the matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ *has trace* $\alpha + \delta = 2$. *If we exclude the case where* $A = \mathrm{I}_2$ *(compare Example 2), the determinant* $D = \alpha\delta - \beta\gamma$ *provides a complete isomorphism invariant for this Lie algebra.*

Proof. We consider a 3-dimensional Lie algebra $\mathfrak{g}$ which is not unimodular and define $\mathfrak{u} = \{u \in \mathfrak{g},\ \mathrm{tr}(\mathrm{ad}_u) = 0\}$. Since $\mathfrak{g}$ is nonunimodular then $\mathfrak{u}$ is 2-dimensional unimodular ideal and hence it is commutative. Choose $e_1 \in \mathfrak{g}$ such that $\mathrm{tr}(\mathrm{ad}_{e_1}) = 0$. Since $\mathfrak{u}$ is commutative, the linear transformation

$$L(u) = [e_1, u]_{\mathfrak{g}}$$

from $\mathfrak{u}$ to itself, with trace 2, is independent of the particular choice of e_1.

If L maps each vector to a multiple of itself, then we are in the special case of Example 2 (and in fact L must be the identity). otherwise, the determinant D of L provides a complete isomorphism invariant. For choosing e_2 so that the vector e_2 and $L(e_2) = e_3$ are linearly independent, the conditions $\mathrm{tr}(L) = 2$ and $\det(L) = D$ imply

$$L(e_2) = e_3 \quad \text{and} \quad L(e_3) = -De_2 + 2e_3.$$

Thus the bracket product operation is uniquely determined. □

Curvature properties can be described as follows. Consider a Lie group as in Lemma 1.7.2.

Theorem 1.7.1 *If the determinant* $D \leq 0$ *then every left invariant metric has Ricci form of signature* $(+, -, -)$. *But if* $D \geq 0$ *the signature* $(0, -, -)$ *is also possible, and if* $D > 0$ *the signature* $(-, -, -)$ *is also possible. In fact, for* $D > 0$ *there exists a left invariant metric of strictly negative sectional curvature and for* $D > 1$ *there exists a left invariant metric of constant negative sectional curvature. In all cases the scalar curvature is strictly negative.*

Proof. See [13]. □

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